Nonlinear (in)stability of some special solutions to the Lorentzian Constant Mean Curvature Flow

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Based on

`arXiv:1310.5606` *(with R. Donninger, J. Krieger, and J. Szeftel)*

`arXiv:1404.0223`

Dynamics of extended bodies

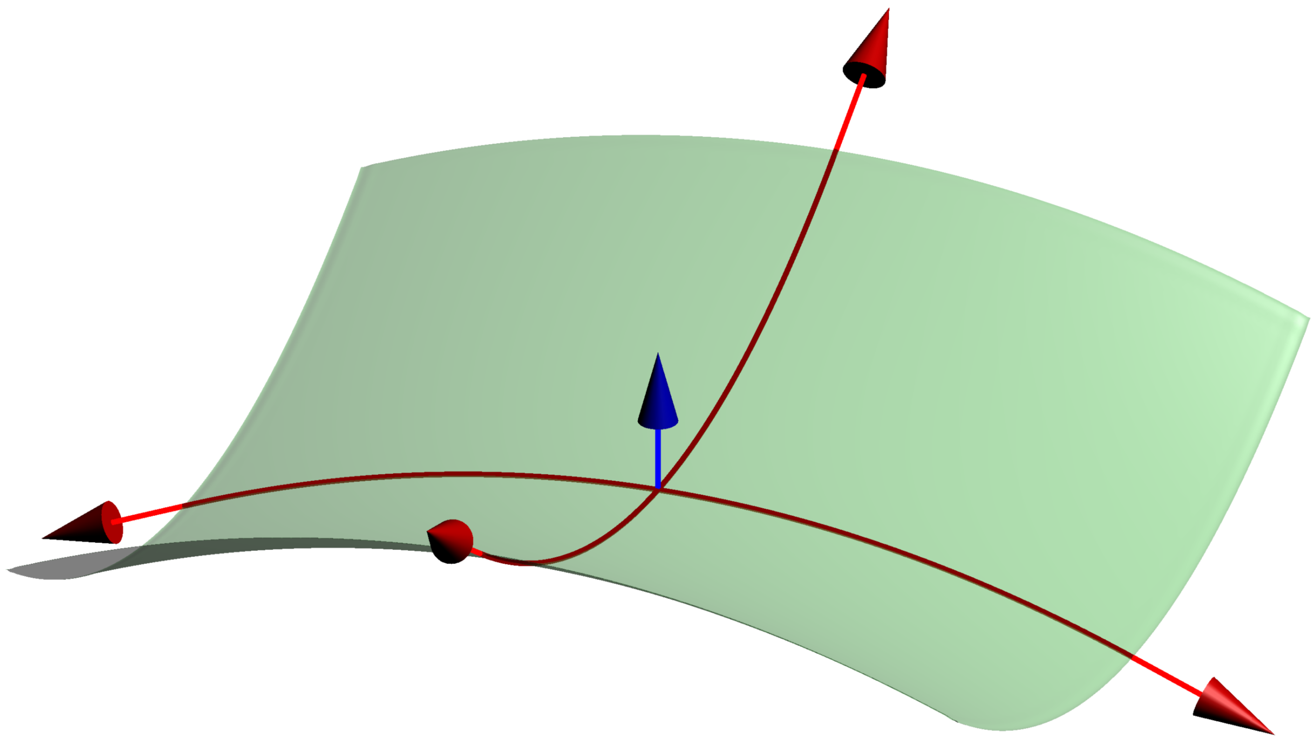
	Point particle	*Extended object*
Geometry	world line	world sheet
Newtonian	$\vec{F} = m\vec{a}$	?
Relativistic	$\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}$	?

Application to HEP:

classical field theory of strings and membranes

Strings in linear regime: D'Alembert (1747) *(The first PDE!)*

Nonlinear theory: surface tension and mean curvature



Young (1805), Laplace (1806), Gauss (1830)

Fluid interface: $\Phi \stackrel{\text{def}}{=} \Delta\text{pressure} \propto \mathrm{MC}$

Dynamics of extended bodies

	Point particle	*Extended object*
Geometry	world line	world sheet
Newtonian	$\vec{F} = m\vec{a}$	$(\Phi - MC)\vec{n} = \mu\vec{a}$
Relativistic	$\vec{F} = \frac{\mathrm{d}}{\mathrm{d}t}\vec{p}$	$\vec{\Phi} = \vec{MC}_{\text{space-time}}$

Particle: static $\Leftrightarrow$ no force $\Leftrightarrow$ flat/linear

Extended: <u>hyperplane</u> (static + no force + flat)

<u>minimal submanifold</u> (static + no force + curved)

<u>bubbles</u> (static + force + curved)

External force

No force $\Leftrightarrow MC = 0$

Full theory: coupled external fields
(fluids, electromagnetism, gravity, etc.)

Codim. 1 case: can scalarise

Lorentzian Constant Mean Curvature Flow

Relativistic dynamics of codimension-one extended test bodies subject to constant external normal force

Model: Hypersurfaces in Minkowski space $\mathbb{R}^{1,d+1}$ with

- Lorentzian induced metric
- MC scalar $\equiv 0$ or $\pm(d+1)$ (rescale)

Study Cauchy problem ("flow")

Cauchy problem with data at fixed time

Initial data:

- position: $\Sigma \subset \mathbb{R}^{d+1}$ a (smooth) hypersurface
- momentum: $p : \Sigma \to \mathbb{R}^{d+1}$ a vector field

Solution: $M \subset \mathbb{R}^{1,d+1}$ a hypersurface s.t.

- M is CMC (with requisite value)
- $M \cap \{t = 0\} = \{0\} \times \Sigma$
- for $q \in M \cap \{t = 0\}$, the tangent space $T_q M = T_q \Sigma \oplus \left(1, \frac{p(q)}{\sqrt{1+|p(q)|^2}}\right)$

Note $T_q M$ is Lorentzian as the speed $\left|\frac{p}{\sqrt{1+|p|^2}}\right| < 1$

Equation in local coordinates

Neighbourhood $N_q \subset M$ is a graph over tangent hyperplane:

$$N_q = \left\{ q + y + \phi(y)n \,\middle|\, y \in T_qM,\ n \perp T_qM \right\}$$

Parametrise $T_qM \cong \mathbb{R}^{1,d}$ with Minkowskian coordinates $\{y^\mu\}_{\mu=0,\dots,d}$:

$$\frac{\partial}{\partial y^\mu}\left(\frac{\eta^{\mu\nu}\partial_\nu\phi}{\sqrt{1+\eta(\partial\phi,\partial\phi)}}\right) = C$$

where C is MC, and $\eta = \mathrm{diag}(-1, 1, \dots, 1)$ is Minkowski metric

QNLW $\Leftrightarrow \eta(\partial\phi, \partial\phi) > -1 \Leftrightarrow M$ is Lorentzian

Local well-posedness

Theorem 1. *Smooth data* $\Longrightarrow \exists!$ *maximal Cauchy development*

Proof. Spatially localized LWP for QNLW + finite speed of propagation + locally finite cover $\square$

Theorem 2. Σ *compact or complete asymptotically graphical* + p *uniformly bounded* $\Longrightarrow$ *lower bound on time of existence by* $\|data\|$

Proof. Finite cover + QNLW breakdown criterion
$[\eta(\partial\phi, \partial\phi) > -1$ and $\left\|\partial\phi\right\|_{W^{1,\infty}} < \infty]$ $\square$

Global problems

Do singularities form?

No: global existence	Yes: finite-time blow-up
• Stability • Asymptotic behaviour	• Stability • Asymptotic profile • Structure of singularity • Weak extension

Today's talk:

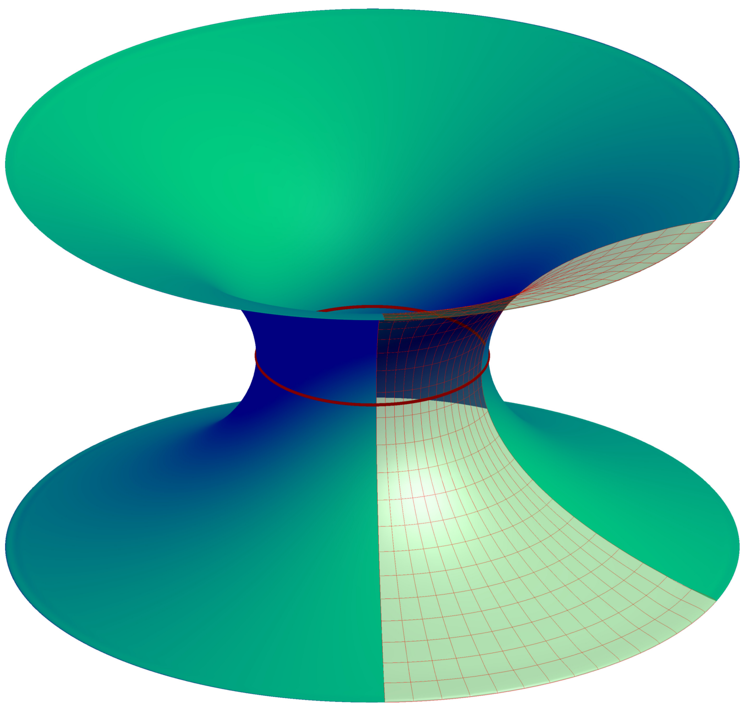
Stability of some special solutions known to exist globally in time

Special solutions; MC $= 0$

Hyperplane:	"Catenoid":
$\mathbb{R}^{1,d} \times \{0\}$	$\mathbb{R} \times \Sigma$ with $\Sigma \subset \mathbb{R}^{d+1}$
	Σ hypersurface of revolution
	$\begin{cases} r'(y) = \operatorname{sgn}(y)\sqrt{1 - r^{-2(d-1)}} \\ z'(y) = r^{-(d-1)} \end{cases}$
Static	Static
Totally geodesic	Negatively curved
Flat	Asymptotically flat

Spatial slices of "catenoids"

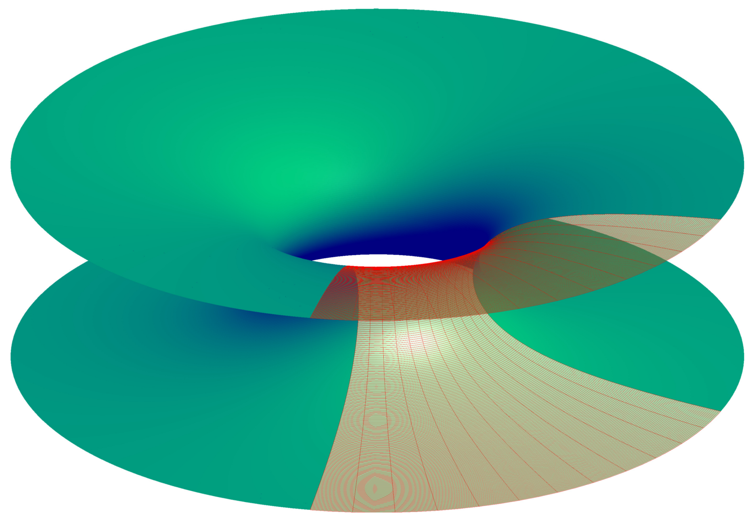
$d = 2$:



$z = \sinh^{-1}(y)$

$r = \sqrt{1 + y^2}$

$d = 3$ (some dim's suppressed):

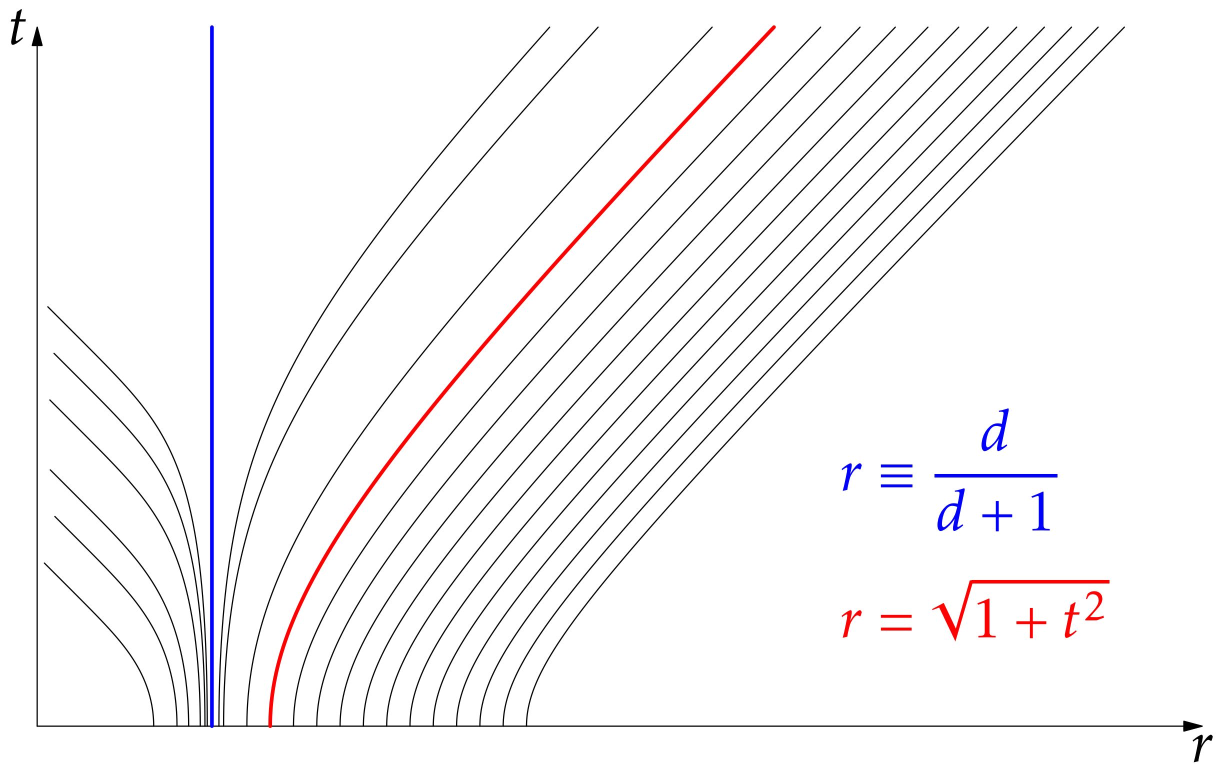


bounded in z

$r \approx 1 + |y|$

Special solutions; MC $= (d+1) > 0$

Rotational symmetry: $r = r(t)$, $\quad \left(\dfrac{r'}{\sqrt{1-(r')^2}} \right)' = (d+1) - \dfrac{d}{r\sqrt{1-(r')^2}}$



Special solutions; MC $= (d+1) > 0$

Einstein cylinder:	de Sitter space:
$\{r = d/(d+1)\}$	$\{r = \sqrt{1+t^2}\}$
$\cong \mathbb{R} \times \mathbb{S}^d$	$= \{\eta(x,x) = 1\}$ (pseudosphere)
Static	Expanding

Positively curved

Compact spatial section

Question

For each of the three solutions

- hyperplane
- “catenoid”
- de Sitter

<u>Is it stable under “small perturbations” w.r.t. the Lorentzian CMC flow?</u>

(Work in progress: Einstein cylinder.)

But... trivial instability?

Actions of the Poincaré group, *except*

- Hyperplane: tangential translations and boosts
- Catenoid: time-translations and co-axial rotations
- de Sitter: Lorentz group

(MC = 0 also dilations)

How do we properly "ignore" these trivial instabilities (TIs)?

A Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	TI + ?		
Cat.	TI + ?		
dS	TI + ?		

Hyperplane

- Perturbations are graphs
- General theory of quasilinear wave equations
- Kill TIs by spatial decay of data: compact support, weighted Sobolev spaces etc
- No other instability
- Dispersion $\implies$ decay

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	$\cancel{\mathrm{TI}}$ $(\infty)+\emptyset$	dispersion $t^{-\frac{d-1}{2}}$	
Cat.	TI + ?		
dS	TI + ?		

Null condition

When $d > 3$, $t^{-(d-1)/2}$ integrable $\implies$ stability

$d \le 3$, quadratic obstruction
$d \le 2$, cubic obstruction
$d = 1$, no dispersion

Quadratic NC: Christodoulou (1986) and Klainerman (1986)
Cubic NC: Alinhac (2001) + earlier weaker versions

Idea: decay for $\partial\phi$ not isotropic;
"tangential" components gain t^{-1}
only one bad "transverse" direction
$\implies$ can exploit good product structure

Brendle (2002) for $d \ge 3$ and Lindblad (2004) for $d \ge 1$(!)

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	~~TI~~ $(\infty) + \emptyset$	dispersion $t^{-\frac{d-1}{2}}$; NC	stable
Cat.	TI + ?		
dS	TI + ?		

Equation of motion: normal-graph gauge

Describe perturbation as graph in normal bundle:

- M — a special solution $\qquad$ g — induced Lorentzian metric
- n — unit normal v.f. $\qquad$ k — 2nd fund. form
- Perturbed solution

$$\tilde{M} = \{\, q + \phi(q) n \;\big|\; q \in M \,\}$$

$\phi : M \to \mathbb{R}$ — the "height"; solves

$$\Box_g \phi + (k : k)\phi = \text{nonlinearity}$$

$k : k$ — double contraction $\qquad$ $\Box_g$ — wave operator (M, g)

Aside: linearised equations

Linearised perturbation equations of non-trivial solutions are typically not the geometric wave equation

Background $\implies$ lower order terms

Example: Regge-Wheeler equations in general relativity

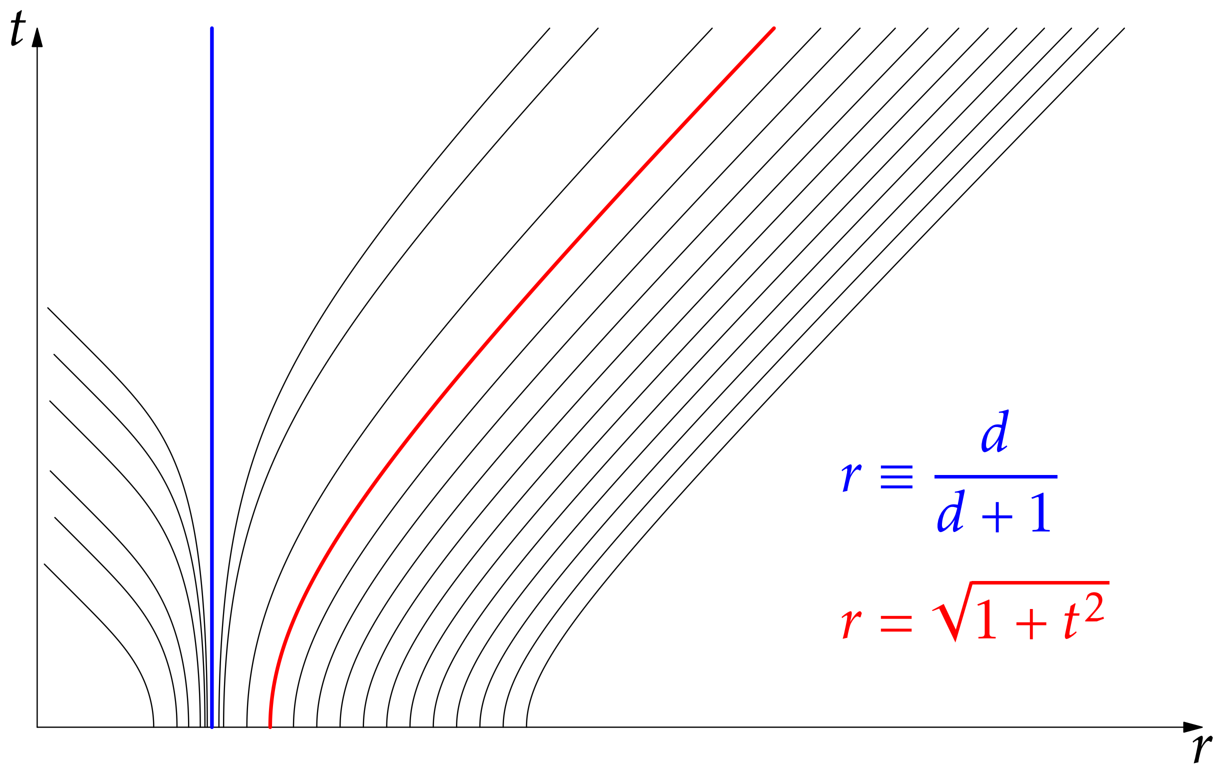
Linear instability

Linearised equation: $\quad \Box_g \phi + (k:k)\phi = 0$

	Hyp.	Cat.	dS
$k:k$	0	$\frac{d(d-1)}{r^{2d}}$	$d+1$

$(k:k) > 0 \Leftrightarrow$ attractive potential
exponentially growing modes
geometric origins

Instability in spherical symmetry, $MC = d + 1$



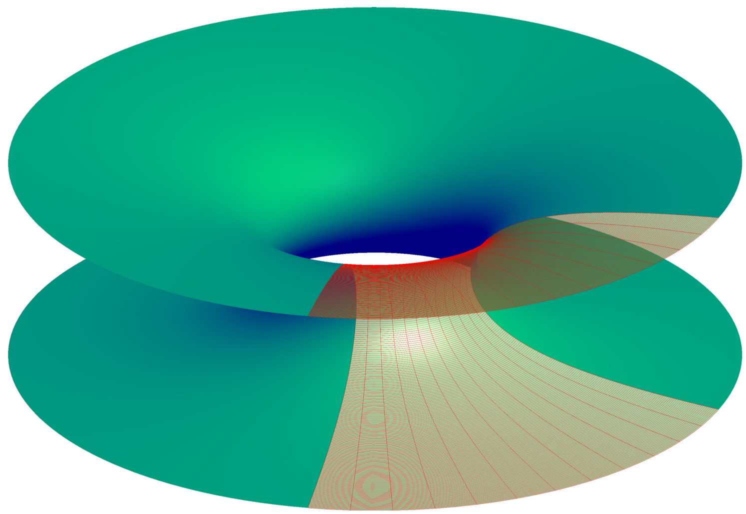
Clear for EC, but de Sitter? (Return to this later)

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	~~TI~~ (∞)	dispersion $t^{-\frac{d-1}{2}}$; NC	stable
Cat.	TI + mode + ?		
dS	TI + mode(?) + ?		

Catenoid: asymptotic flatness

Decay at ∞ kills some of TIs; except translations orthogonal to z



Far away from "throat", $t^{-(d-1)/2}$ dispersion holds generally

Catenoid: null condition

Given solution to QNLW, whether NC holds for perturbation equations depends on solution

<u>Example</u>: fluid models and formation of shock
(Christodoulou 2007; Holzegel-Klainerman-Speck-Wong 2014)

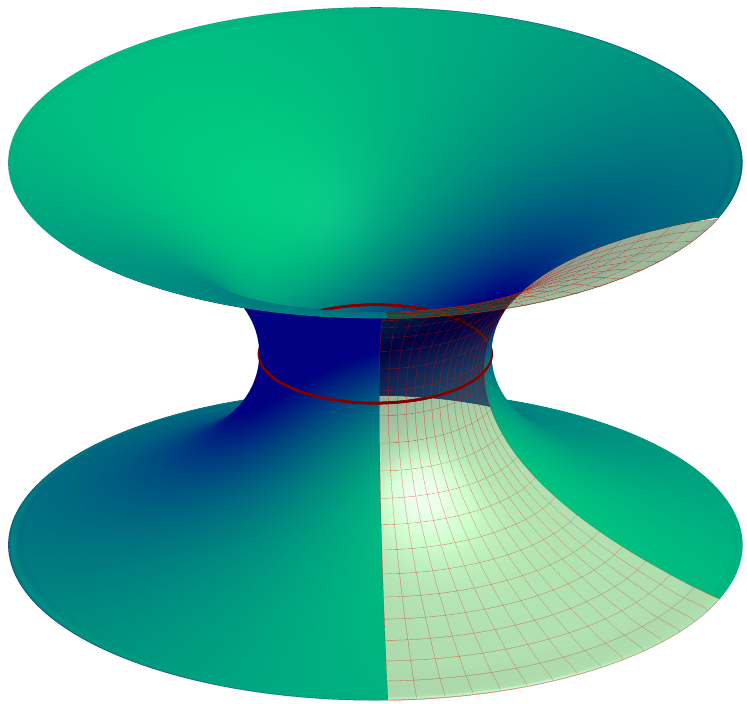
<u>Exception</u>: Lorentzian CMC flow
(characterisation among "fluids")

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	~~TI~~ (∞)	dispersion $t^{-\frac{d-1}{2}}$; NC	stable
Cat.	TI (some killed at ∞) + mode + ?	ext. disp. + NC	
dS	TI + mode(?) + ?		

Catenoid: trapping

closed geodesics at the throat $\implies$ trapped null geodesics



wave packets $\implies$ derivative loss in decay estimates expected

major obstacle, work in progress

Catenoid: symmetry

Axial symmetry assumption

- Kills remaining TI
- Removes trapping

Problem still non-trivial

- Linear instability
- How to capture dispersion near the throat?

Catenoid: centre stable manifold

y denotes the radial unit-length coordinate on the catenoid
$y = 0$ at the throat.

For $\phi = \phi(y)$ a radial C_0^∞ function on a catenoid we define the weighted Sobolev norm

$$\|\phi\|_{X^k} = \sum_{j=0}^{k} \left\|(1+y^2)^{k/2} \partial_y^j \phi(y)\right\|_{L^1 \cap L^2(r^{(d-1)/2}\,\mathrm{d}y)}$$

Let ϕ_{gs} denote the exponential growing mode

Catenoid: centre stable manifold

Theorem 3 (DKSW 2013)**.** *There exists* $N \gg 0$ *and* $\delta > 0$ *such that for every* $(\phi_0, \phi_1) \in C^\infty_{0,rad}(Cat)$ *verifying*

$$\left\|\phi_0\right\|_{X^{N+1}} + \left\|\phi_1\right\|_{X^N} < \delta,$$

there exists $\alpha = \alpha(\phi_0, \phi_1)$*, with Lipschitz continuous dependence on* (ϕ_0, ϕ_1) *w.r.t.* $X^{N+1} \times X^N$*, such that the modified initial perturbation*

$$(\phi_0 + \alpha\phi_{gs}, \phi_1)$$

leads to a solution existing globally in forward time and converges asymptotically to 0 at the rate $\sup_y |\phi(t,y)| \lesssim (1+t)^{-\frac{1}{2}}$.

Remark:

Proved for $d = 2$, but same technique carries identically to $d > 2$

Catenoid: centre stable manifold

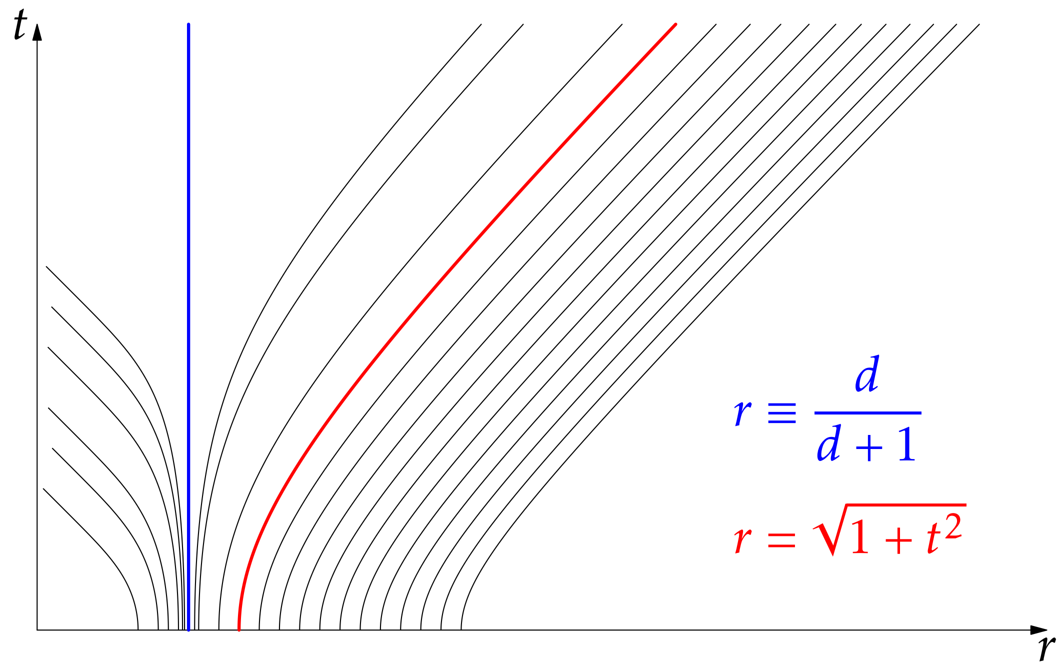
Key ideas:

- Orthogonally decompose $\phi = h(t)\phi_{gs} + \varphi$
- Linear decay (incl. int.) for φ using distorted Fourier transform
- $h(t)$ solves "ODE"
- Bootstrap:
  - first estimates for φ; use null condition
  - IBP for top order terms
  - then estimates for h; use one-dimensionality of unstable directions to modify α if needed

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	~~TI~~ (∞)	dispersion $t^{-\frac{d-1}{2}}$; NC	stable
Cat.	~~TI~~ (sym/∞) + ~~trap~~ (sym) + mode	dispersion + NC	axi-sym. codim 1 stable
dS	TI + mode(?) + ?		

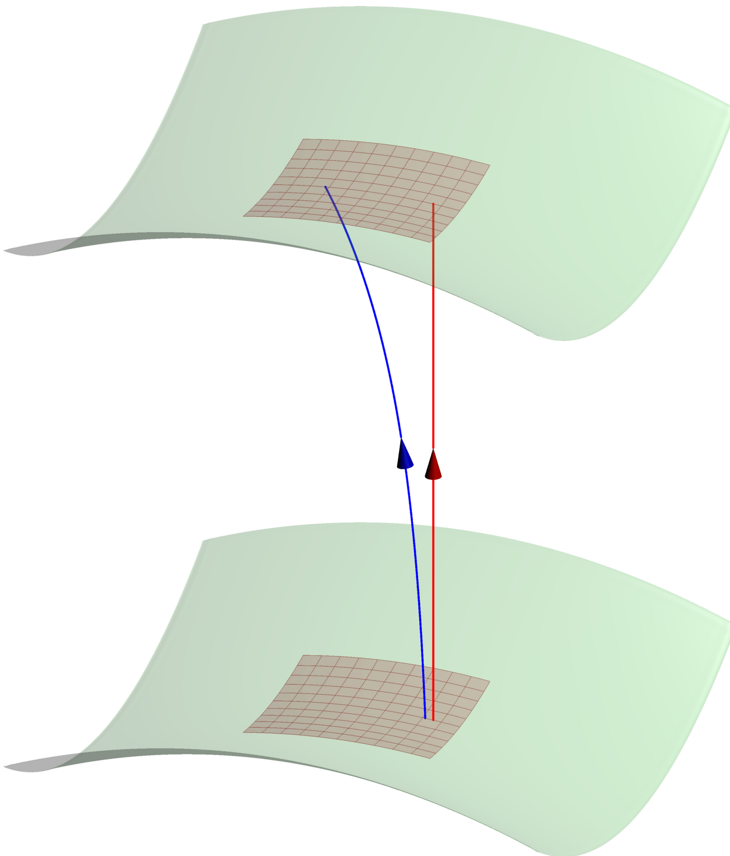
de Sitter: mode instability?



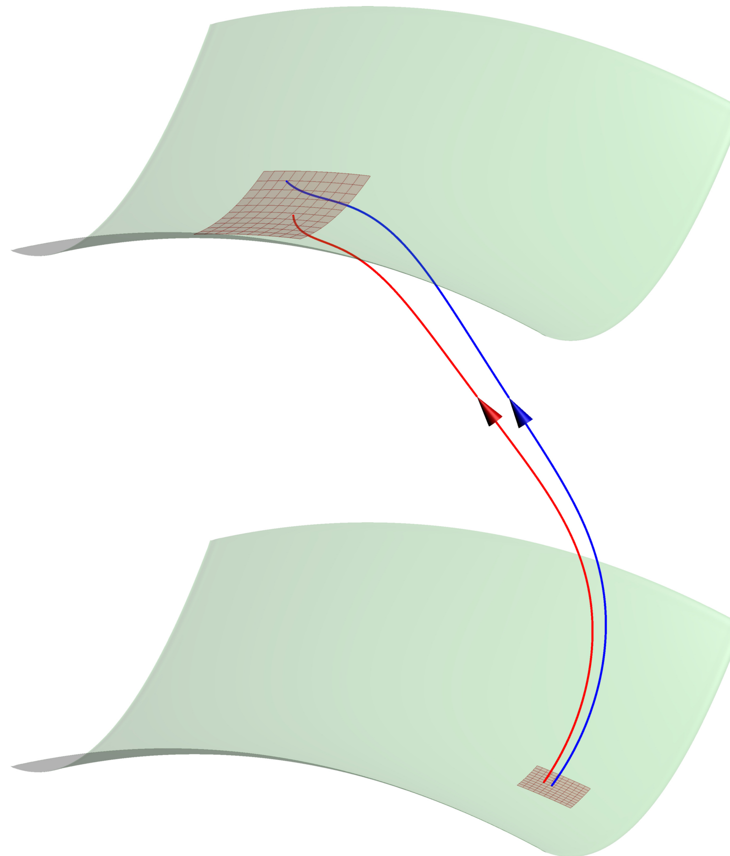
Rel. to normal bundle: exp. growth in proper time
linear growth in ambient time t

Rel. to ambient coordinate system: bounded!

Deceptive linear stability analysis



Static: same function space $\forall t$



Non-static: <u>how to compare</u>?

de Sitter: . . . maybe just a bad gauge?

Rel. to normal bundle, the translation TI also “exp. growth”

In spherical symmetry, all solutions approach light-cones:
unique asymptotic profile

Can we modulate? (Can’t kill TI: no ∞, also time translation)

Interlude: modulation theory

Basic philosophy: Stability of a family of solutions

- Represent family as manifold in function space
- Decompose EoM to ODE on manifold + orthogonal PDE
- Show PDE decays

Soliton dynamics for NLW, NLS
Due originally to M.I. Weinstein (1985, 1986)

de Sitter: modulation?

$$
\begin{array}{rcl}
\text{Solution} & & \text{Asymptotics} \\
\Downarrow & & \Uparrow \\
\text{dS w/ time-dependent} & \Longrightarrow & \text{convergence in time} \\
\text{space-time translation} & & \\
+ & & + \\
\text{perturbations} & \Longrightarrow & \text{decay}
\end{array}
$$

Works in spherical symmetry (single asymptotic profile)

de Sitter: modulation insufficient

Theorem 4 (W 2014)**.** *The linearised equation has infinitely many exponential growing modes.*

Theorem 5 (W 2014)**.** *For full nonlinear problem,* $\exists$ *arbitrarily small perturbations whose global solutions cannot converge to any space-time translation of dS.*

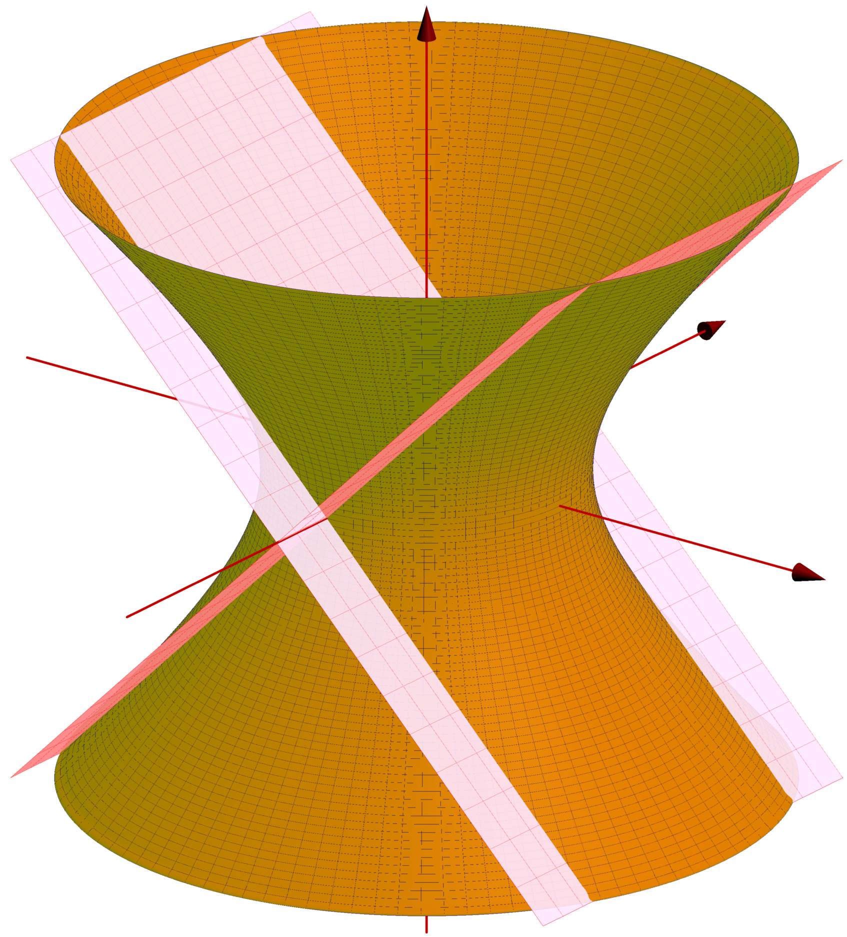
de Sitter: cosmological horizon

space expanding w/ speed > c

separate asym. regimes
- Minkowski: only one
- dS: entire sphere

many "time-like" infinities; different asymptotics

amplifies TI to ∞-dim



de Sitter: cosmological horizon

Already seen in GR:

- Linear wave on asymp. dS manifolds
- Non-linear stability of FLRW geometries
  "freezing-in" perturbations
  (H. Ringstrom, J. Speck, I. Rodnianski, etc.)

Expansion $\implies$ spatial derivatives decay exp. faster
(Mink: polynomial)

<u>True even for "our" linearised equation (w/ potential)</u>

de Sitter: spatial-local stability

Theorem 6 (W 2014). *A sufficiently small (in H^N, $N > d+3$) perturbation of de Sitter initial data leads to*

- *a solution that exists global in time, s.t.*
- *for every ideal point ω in future time-like infinity ($\cong \mathbb{S}^d$) there exists $(\tau, \xi) \in \mathbb{R}^{1,d+1}$ such that in the past domain of dependence of ω, the solution converges to dS translated by (τ, ξ). The mapping $\omega \mapsto (\tau, \xi)$ is Lipschitz, with global bound by the size of initial perturbation.*

de Sitter: main ideas

Study second fundamental form instead of "graph"

- intrinsic geometry vs. extrinsic representation
- objects at the level of "derivative" so decay
- equations of motion: Gauss-Codazzi
- GR analogy: Bianchi for Weyl vs. wave for metric

de Sitter: main ideas

IGM: inverse Gauss map gauge

- compare perturbation and original via Gauss map
  - GM in $\mathbb{R}^{1,d+1}$ takes values in dS!
- space-time dependent modulation
- quasilinear div-curl system for Sym-2 field on dS
  - develop vector field method for such systems
- GR analogy: choice of foliation in stab. of Mink.

de Sitter: main ideas

Vector field method for quasilinear div-curl system

- energy-momentum $\Longrightarrow$ Bel-Robinson type tensor (constant coeff. case by Brendle 2002)
- commutators: ambient rotations = intrinsic translations (note: t-weighted)
- multipliers: unit time vector (note: not Killing!)
- t-weighted energy estimates (note: weight necessary due to expansion)

Chart

	Instability mech.	*Stability mech.*	*Result*
Hyp.	~~TI~~ (∞)	dispersion $t^{-\frac{d-1}{2}}$; NC	stable
Cat.	~~TI~~ (sym/∞) + ~~trap~~ (sym) + mode	dispersion + NC	axi-sym co-dim 1 stable
dS	~~TI + 'mode'~~ (IGM)	exp. der. decay from expansion	spatial-local stability

Thank you!