TIMELIKE MINIMAL SUBMANIFOLDS OF MINKOWSKI SPACES

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Abstract. Time-like minimal submanifolds arise as models of cosmic strings and membranes. Their definition, requiring the mean curvature vector to vanish identically, can be expressed as a system of geometric PDE. The further requirement that these submanifolds are timelike (in other words, the induced metric from the ambient Minkowski space is Lorentzian) gives these equations the character of hyperbolic PDE. From the analytic point of view, it is natural to think of this as an evolution equation and study the associated Cauchy problem. In this talk we will review aspects of this problem and present some recent developments.

1. The main players

Let us set the stage by introducing the notations. Throughout we will consider $\mathbb{R}^{1,N}$ being the $(1+N)$-dimensional Minkowski space, equipped with the Minkowski metric $\eta = \mathrm{diag}(-1,1,1,\dots,1)$. We let M denote some $(1+d)$-dimensional smooth manifold that is realized as a submanifold of $\mathbb{R}^{1,N}$ by the *immersion* ϕ. Throughout we shall assume that the immersion ϕ is *time-like*, in that the pullback metric $g := \phi^*\eta$ on M has Lorentzian signature.

The standard coordinate functions on $\mathbb{R}^{1,N}$ will be denoted $\{x^0,\dots,x^N\}$. The induced time function on M is $t := x^0 \circ \phi$. We label the level sets of t by Σ_t, which are d-dimensional manifolds with induced Riemannian metric h.

The main condition we will impose on ϕ is *minimality*, which we take to be equivalent to the induced, vector-valued mean curvature vanishing identically. It is worth noting that we do not actually require the immersion to be area minimizing, but following the fine tradition in Riemannian geometry only require it to be stationary points of the area functional under compactly supported perturbations. This condition originated in the physics literature as a model of extended test objects. Much in the same way that time-like geodesics (which can be interpreted as curves with zero extrinsic, or geodesic, curvature) in Lorentzian manifolds represent world lines of free-falling particles, the time-like zero-mean-curvature submanifolds of higher dimensions in a Lorentzian manifold are taken to represent the world *sheets* of extended test objects evolving under no external force. And in the case where external forces are present, the forces are assumed to interact with these objects through prescription of mean curvature; see [4, 8, 14, 24] for discussions of the physics surrounding these objects. That time-like minimal submanifolds can be a good model for extended particles has also been verified in [9, 20], where it was shown that for certain semilinear wave equations, one

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can construct solutions whose energy is concentrated around time-like minimal hypersurfaces.

Aside from the physical motivation, it appears that the study of the equations associated to such minimal immersions may be relevant to understanding singularity formation in general quasilinear wave equations, when restricted to "non-shock-type" singularities. In section 3 of this talk we will briefly explain the intuition behind this statement, which also serves to motivate the result that will be described in section 5. Aside from the material presented in section 3, much of the rest of this contribution can be found, with more technical detail, in the work [26].

2. The equation and its hyperbolicity

The following proposition is well-known, at least in the Riemannian case. It can be proved by a direct computation in normal coordinates.

Proposition 1. *Let $(\tilde{M}, \tilde{g})$ be a smooth pseudo-Riemannian manifold. If M is a smooth manifold and $\phi : M \to \tilde{M}$ is an immersion with nondegenerate pull-back metric $g := \phi^* \tilde{g}$, then the following are equivalent:*

(i) ϕ is minimal;
(ii) ϕ is a harmonic map of $(M, g) \to (\tilde{M}, \tilde{g})$.

Now, returning to our setup where $\tilde{M} = \mathbb{R}^{1,N}$ with the Minkowski metric η, an immediate consequence (due to the flatness of Minkowski space) is that, writing $\{x^A\}_{A\in\{0,\ldots,N\}}$ for the standard rectilinear coordinates on $\mathbb{R}^{1,N}$, our minimal immersion ϕ must then satisfy the equation

$$\Box_g (x^A \circ \phi) = 0, \tag{1}$$

where $\Box_g$ is the Laplace-Beltrami operator for the pseudo-Riemannian manifold (M, g).

While at first glance (1) suggests that the components of ϕ satisfy a system of quasidiagonal wave equations, and therefore has a locally well-posed Cauchy problem, a deeper investigation reveals that due to the dependence of $g = \phi^* \eta$ on the derivatives $\mathrm{d}\phi$, the system as written is in fact not quasidiagonal. Furthermore, local uniqueness in fact fails for the system (1). Observe that if $\phi : M \to \mathbb{R}^{1,N}$ is a minimal immersion, then for any diffeomorphism $\zeta : M \to M$ the composition $\phi \circ \zeta$ is also a minimal immersion.

However, that diffeomorphism invariance provides an obstruction to local uniqueness is a common feature in geometric partial differential equations, with two of the most well-known examples being Einstein's equations in general relativity and the equations for Ricci flow. The solution is to fix a parametrization of the manifold M by imposing a gauge condition. In the relativity literature many different gauge conditions have been explored; in Ricci flow the most common is the so-called De Turck trick.

To give an example of gauge-fixing in the context of time-like minimal immersions, consider the (dynamic) parametrization of M by *harmonic coordinates* $\{y^\nu\}_{\nu\in\{0,\ldots,d\}}$ relative to the induced metric g. The gauge condition is that

$$\Box_g y^\nu = 0 \tag{2}$$

or equivalently

$$\sum_{\rho,\sigma\in\{0,\ldots,d\}} \Gamma^{\nu}_{\rho\sigma}(g^{-1})^{\rho\sigma} = 0$$

where $\Gamma^{\nu}_{\rho\sigma}$ is the Christoffel symbol of g in the y-coordinates. Imposing this coordinate condition, the minimality condition (1) reduces to

$$\sum_{\rho,\sigma\in\{0,\ldots,d\}} (g^{-1})^{\rho\sigma}\partial^2_{\rho\sigma}(x^A\circ\phi) = 0 \tag{3}$$

where ∂ is the coordinate partial derivative relative to the y-coordinate system. In this formulation the equation becomes bona fide system of quasidiagonal quasilinear wave equations, and thus inherits the good local well-posedness properties common to all such systems.

Notice that equation (1) implies that the ambient coordinate functions x^A restrict to harmonic functions on M. The assumption that ϕ is a immersion implies locally one can isolate $1+d$ of the x^A to form a nondegenerate coordinate system y^ν of M; by construction this coordinate system is harmonic. In this coordinate system $\phi(M)$ can be described as a graph over the corresponding subspace. The equations of motion then take the familiar divergence form

$$\sum_{\rho,\sigma\in\{0,\ldots,d\}} \frac{\partial}{\partial y^\rho}\frac{\eta^{\rho\sigma}\tilde{\phi}_{,\sigma}}{\sqrt{1+\sum_{\mu\nu}\eta^{\mu\nu}\langle\tilde{\phi}_{,\mu},\tilde{\phi}_{,\nu}\rangle}} = 0 \tag{4}$$

where $\tilde{\phi}:\mathbb{R}^{1,d}\supset U\to\mathbb{R}^{N-d}$ is such that locally we can identify $\phi(M)$ with $(y,\tilde{\phi}(y))$. The inner product $\langle\cdot,\cdot\rangle$ appearing in the denominator of (4) is the Euclidean inner product on $\mathbb{R}^{N-d}$.

3. Null condition and consequences

In the graphical formulation (4), the induced metric takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \langle\tilde{\phi}_{,\mu},\tilde{\phi}_{,\nu}\rangle. \tag{5}$$

Writing

$$\Upsilon^{\beta}_{\alpha} = \sum_{\sigma=0}^{d}\langle\tilde{\phi}_{,\alpha},\tilde{\phi}_{,\sigma}\rangle\eta^{\sigma\beta},$$

the series expansion near $\tilde{\phi}\equiv 0$ of the inverse metric is

$$(g^{-1})^{\mu\nu} = \sum_{\rho,\alpha,\beta,\ldots\in\{0,\ldots,d\}} \eta^{\mu\rho}\Big(\delta^{\nu}_{\rho} - \Upsilon^{\nu}_{\rho} + \Upsilon^{\alpha}_{\rho}\Upsilon^{\nu}_{\alpha} - \Upsilon^{\alpha}_{\rho}\Upsilon^{\beta}_{\alpha}\Upsilon^{\nu}_{\beta} + \ldots\Big). \tag{6}$$

This means that, after writing (4) in the form (3), it is immediately obvious that the equations of motion for time-like minimal submanifolds satisfy Klainerman's null condition [15, 16], as there are no quadratic nonlinearities. Furthermore, the cubic nonlinearities are of the form

$$\langle\tilde{\phi}_{,\mu},\tilde{\phi}_{,\nu}\rangle\eta^{\mu\rho}\eta^{\nu\sigma}\partial^2_{\rho\sigma}\tilde{\phi}$$

which one can easily check to verify the *cubic* null condition [1, 2]. Thus it follows from the general machinery of quasilinear wave equations that the Cauchy problem for time-like minimal submanifolds, with the dimension $d\geq 2$, is *globally* well-posed for all sufficiently small (in certain weighted Sobolev spaces) initial data perturbations of the trivial hyperplane solution; see also the proofs by Brendle [5] and Lindblad [18] which are specifically adapted to the time-like minimal hypersurface case, as well as the extension by Allen, Anderson, and Isenberg [3] to general codimensions.

The null conditions are structural conditions on systems of partial differential equations that capture "extra cancellations". Take the example of wave equations: the linear wave equation on $\mathbb{R}^{1,d}$ are known to decay generically like $(1+|t|)^{-(d-1)/2}$. If one were to try to solve a nonlinear wave equation by iteration, for $d > 3$, the coefficients of the perturbation is then expected to decay at an integrable rate, which allows the iteration to converge. For $d \leq 3$, however, $(1+|t|)^{-(d-1)/2}$ are not integrable, and one can in fact show that small data global wellposedness is often false [11, 13]. The $(1+|t|)^{-(d-1)/2}$ decay for solutions is however only generic, and certain combinations of derivatives are known to decay faster (for example, angular derivatives in polar coordinates). Roughly speaking, for equations satisfying the null conditions, every quadratic (and also cubic when $d = 2$) nonlinear term can be factored so that at least one factor enjoys better decay, thereby guaranteeing that in the iteration scheme, the perturbed coefficients are integrable in time.

An interesting aspect of Lindblad's analyses [18] is that the same global well-posedness result also holds for the $d = 1$ case, if one restricts to *compactly supported* initial perturbations. In the $d = 1$ case solutions to the linear wave equation has no natural decay rate, and if one were to follow the same heuristic argument as in the cases $d = 2, 3$ described above, one would need the equation to exhibit some sort of "infinite-order" null condition (which we note is already suggested by the expression (6)). Lindblad's analysis instead rests on Christodoulou's conformal compactification method [6] and hence requires compact support of the initial data. The "infinite-order null condition", for the time-like minimal surface equation, can be exploited to show that relative to a dynamic, geometric coordinate system ("double null coordinates"), the equation significantly simplifies. (This is also related to the fact that Laplace-Beltrami operators are conformally invariant when the total space-time dimension is 2.) Using this the speaker was able to extend the global wellposedness result to data with much weaker decay and integrability assumptions [27].

The aforementioned "infinite-order" null condition is special to (4); this is in contrast to the typical appearance of the classical null condition of Klainerman in small data problems in physical systems of partial differential equations. Indeed, for general *Lorentz invariant* field theories defined over a fixed Lorentzian manifold (such as the dynamics of homogeneous, isotropic, and perfectly elastic continuous media described in [23] or the types of Lagrangian theory of maps described in [25]), it is a simple exercise to check that the null condition of Klainerman holds. More concretely, we have the following example.

Proposition 2. *Consider the Lagrangian field theory for a scalar field $\phi : \mathbb{R}^{1,d} \to \mathbb{R}$ given by the formal action functional of the form*

$$A = \int L(\sigma)\, \mathrm{d}x \tag{7}$$

where $\sigma := \sum \eta^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ is the trace of the strain tensor, with $L : \mathbb{R} \to \mathbb{R}$ smooth with $L'(0) \neq 0$. Then

(1) The function $\phi \equiv 0$ solves the associated Euler-Lagrange equation.

(2) The linearization of the Euler-Lagrange equation around $\phi \equiv 0$ is the linear wave equation.
(3) The Euler-Lagrange equation satisfies Klainerman's null condition for small data.

Proof. The proposition follows immediately by noting that the Euler-Lagrange equation takes the form

$$\sum_{\mu,\nu\in\{0,\dots,d\}} [L'(\sigma)\eta^{\mu\nu} + 2L''(\sigma)\partial^\mu\phi\partial^\nu\phi]\partial^2_{\mu\nu}\phi = 0. \tag{8}$$

Near $\phi \equiv 0$ the above equation expands to a *cubic* perturbation of the linear wave equation, and the proposition follows. □

Remark 3. Above and henceforth we use the notational shorthand

$$\partial^\mu := \sum_\nu \eta^{\mu\nu}\partial_\nu$$

for the partial derivative.

An emphasis in the previous proposition is that the null condition is satisfied for *small data*. Klainerman's null condition, at the heart, is a "perturbative" condition around the trivial solution; one should think of it as analogous to imposing conditions on the coefficients of a sort of "Taylor expansion" of the equation around $\phi \equiv 0$. Away from $\phi \equiv 0$, of course, Klainerman's null condition does not necessarily give the requisite cancellation for global existence. A spectacular example of this is the case of compressible Euler equations studied by Christodoulou [7]. The irrotational, relativistic, compressible Euler equations admit a Lagrangian formulation in the form (7). However, the physical solutions considered in [7] are perturbations not of the trivial $\phi \equiv 0$ solution, but of the $\phi \equiv t$ solution which correspond to a stationary fluid. And, as was shown in Christodoulou's monograph, these solutions necessarily become singular in finite time.

Another example of the same phenomenon appeared in the work of Miao and Yu [19]: they considered for a cubic-nonlinear wave equation the initial value problem with initial data given by a short pulse ansatz. These initial data are *not small*. And in this case, again, while the Klainerman null condition gives small-data global existence, finite time singularity in fact develop for such large data. In this type of results, the mechanism that drives the singularity formation is that of shock formation due to characteristic crossings. The one dimensional case of this phenomenon is very well known (see e.g. [17] and references there-in; and also [12]); recently the stability of this phenomenon (under higher dimensional perturbations) is shown [22]. In the one-dimensional case the driver for shock formation is the resonance condition known as Lax's "genuine nonlinearity" condition; this is, in some sense, the negation of Klainerman's null condition. If one thinks of Klainerman's null condition as stating, very roughly speaking, that that second order coefficients vanish in a certain "Taylor expansion" about the origin, then Lax's genuine nonlinearity is the requirement that for the corresponding "function" the second derivatives vanish nowhere.

And this brings us back to our discussion of equation (4). In this analogy, (4) represents a quasilinear wave equation for which the "second derivatives" vanish *everywhere*. To be more precise: let us return to Lagrangian field theories of the form (8) given by the action principle (7). Now let $\mathring{\phi}$ be a fixed solution. The principal part of (8) defines an inverse metric

$$\mathring{g}^{\mu\nu} = L'(\mathring{\sigma})\eta^{\mu\nu} + 2L''(\mathring{\sigma})\partial^\mu\mathring{\phi}\partial^\nu\mathring{\phi}. \tag{9}$$

Now, if $\phi = \mathring{\phi} + \psi$ is a perturbed solution, then ψ would solve an equation of the form

$$\sum_{\mu,\nu\in\{0,\ldots,d\}} \mathring{g}^{\mu\nu}\partial^2_{\mu\nu}\psi + \sum_{\mu,\nu,\lambda\in\{0,\ldots,d\}} \mathring{G}^{\mu\nu\lambda}\partial_\lambda\psi\partial^2_{\mu\nu}\psi = \ldots \tag{10}$$

where the $\ldots$ on the right hand side captures all

- cubic or higher quasilinear terms (terms that look like $(\partial\psi)^k\partial^2\psi$ for $k \geq 2$);
- semilinear terms (nonlinear terms that involve $(\partial\psi)^k$ for $k \geq 2$); and
- potential terms (terms linear in $\partial\psi$ with non-trivial coefficients coming from $\mathring{\phi}$).

The coefficients in the expansion, such as $\mathring{g}^{\mu\nu}$ and $\mathring{G}^{\mu\nu\lambda}$, all obviously depend on $\mathring{\phi}$. Thus, in analogy to how Klainerman's null condition is usually stated, we can state the following generalized version of the null condition.

Definition 4. We say that the action principle (7) (or its associated Euler-Lagrange equation (8)) satisfies the *(quasilinear) null condition relative to the background solution* $\mathring{\phi}$ if

$$\sum_{\mu,\nu,\lambda\in\{0,\ldots,d\}} \mathring{G}^{\mu\nu\lambda}\ell_\mu\ell_\nu\ell_\lambda = 0$$

for every covector ℓ_μ satisfying

$$\sum_{\mu,\nu\in\{0,\ldots,d\}} \mathring{g}^{\mu\nu}\ell_\mu\ell_\nu = 0.$$

It is easy to see that Klainerman's null condition for a quasilinear wave equation is, in the above definition, the null condition relative to the solution $\mathring{\phi} \equiv 0$, and Proposition 2 follows because the associated $\mathring{G}^{\mu\nu\lambda} \equiv 0$.

Proposition 5. *Among action principles of type* (7), *the only Lagrangians L for which (a) $\mathring{g}^{\mu\nu}$ is Lorentzian (b) the null condition hold relative to* any *background solution $\mathring{\phi}$ are $L = c\sigma + d$ and $L = \pm\sqrt{c\sigma + d} + e$ where c, d, e are real constants.*

Remark 6. Of the hypotheses: condition (a) guarantees that the associated equation of motion is hyperbolic, and serves to rule out degenerate situations such as $L = c$ for some constant. In terms of the conclusion: The case $L = c\sigma + d$ is the Lagrangian for the linear wave equation. The case $L = \pm\sqrt{c\sigma + d} + e$ is the general form of the Lagrangian for the Lorentzian minimal hypersurface equation (which is usually normalized as $L = 1 - \sqrt{1 + \sigma}$) that is the focus of this talk.

Remark 7. A different version of this result was discussed in [7]. In there it was shown that the principal part of the acoustic curvature vanishes if an only if L is one of the above-described functions. Fundamentally that and the above proposition manifest from the same phenomenon; our description here has the advantage of being shorter and easier to prove.

Proof. By a direction computation plugging in $\phi = \mathring{\phi} + \psi$ into (8), we obtain that the coefficients

$$\mathring{G}^{\mu\nu\lambda} = 2L''(\mathring{\sigma})\eta^{\mu\nu}\partial^\lambda\mathring{\phi} + 4L'''(\mathring{\sigma})\partial^\lambda\mathring{\phi}\partial^\mu\mathring{\phi}\partial^\nu\mathring{\phi} + 4L''(\mathring{\sigma})\eta^{\lambda\nu}\partial^\mu\mathring{\phi}. \tag{11}$$

By (9) the null covector ℓ_μ must satisfy

$$\sum_{\mu,\nu\in\{0,\dots,d\}} L'(\mathring{\sigma})\eta^{\mu\nu}\ell_\mu\ell_\nu = -2L''(\mathring{\sigma})\Big[\sum_{\mu=0}^{d}\ell_\mu\partial^\mu\mathring{\phi}\Big]^2.$$

This implies, by (11), that

$$\sum_{\mu,\nu,\lambda\in\{0,\dots,d\}} L'(\mathring{\sigma})G^{\mu\nu\lambda}\ell_\mu\ell_\nu\ell_\lambda = 4\Big[\sum_{\mu=0}^{d}\ell_\mu\partial^\mu\mathring{\phi}\Big]^3\Big[L'''L' - 3(L'')^2\Big](\mathring{\sigma}).$$

For this to vanish for *any* background $\mathring{\phi}$ and *any* null covector ℓ_μ, we must have either

(1) $\sum_{\mu=0}^{d}\ell_\mu\partial^\mu\mathring{\phi} = 0$; or
(2) L satisfies $L'L''' = 3(L'')^2$.

Let us examine the two cases in detail.

In the first case, since $\mathring{g}^{\mu\nu}$ is Lorentzian, if $\sum\ell_\mu\partial^\mu\mathring{\phi} = 0$ for all $\mathring{g}$-null covectors ℓ, it must be the case that $\partial\phi = 0$, which implies $\mathring{\sigma} = 0$. In other words, in terms of the Lagrangian density function L, the first condition gives no constraint. And further, for generic background solutions $\mathring{\phi}$, the first case cannot be satisfied for all ℓ.

Hence we focus on the second case. The differential equation $L'L''' = 3(L'')^2$ holds if one of

(1) $L' \equiv 0$; this case is degenerate and ruled out by the first condition of the proposition.
(2) $L'' \equiv 0$; this means $L = c\sigma + d$.
(3) or L solves the log differential equation $L'''/L'' = 3L''/L'$, which implies $\pm L'' = c(L')^3$. The non-trivial solution to this is $L' = \pm(c\sigma + d)^{-1/2}$ and hence our proposition is proved.

□

The proposition above establishes the special place occupied by the linear wave equation and (4) (in the hypersurface case $N = d + 1$) as scalar quasilinear wave equations for which the null condition holds relative to any background solution. In this sense one has a clear description of what it means for the equation to satisfy an "infinite-order null condition". A direct consequence of this, then, is that the shock formation mechanism based on intersection of characteristic hypersurfaces that underly both the one-dimensional case and the recent developments *cannot* be used to describe singularity formation for (4). Conversely, the study of the time-like minimal submanifold equation can potentially give some insight to non-shock-type singularity formation mechanisms in generic quasilinear wave equations. The remainder of this talk concerns precisely some results related to singularity formation mechanisms for the time-like minimal submanifold system.

4. Singularities for compact strings

If one were to require the initial data for the evolution be a round sphere, then the equations of motion reduces to an ordinary differential equation for the radius

$$(r')^2 = 1 - cr^{N-1}$$

which can be found by evaluating an elliptic integral and the solution can be seen to shrink to a point in finite time.

Outside of this high level of symmetry, the only previous results concerning singularity formation is for compact cosmic strings: in this case $d = 1$ and the solution M is two dimensional with cross sections diffeomorphic to $\mathbb{S}^1$. Observe that by (1) the immersion map solves a wave equation. Since the Laplace-Beltrami operator is conformally invariant in two dimensions, one can in fact find a parametrization (t, y) of M with y taking values in some rescaled circle, such that ϕ in fact solves the linear wave equation on the cylinder

$$-\partial^2_{tt}\phi + \partial^2_{yy}\phi = 0. \tag{12}$$

Remark 8. This is essentially the analogue of the Weierstrass representation formula for minimal surfaces, translated to our Lorentzian setting.

Now, the wave equation (12), being linear, admits solutions for all times t, and the solution is as smooth as the initial data. (In fact the fundamental solution is quite simple to write down.) However, the solution need not represent an *immersed* submanifold of $\mathbb{R}^{1,N}$: in the case the initial data is a round circle in the plane, the solution is given by rescaled round circles with the scaling equal to $\sin(t)$. In other words, $\partial\phi$ could be singular. Particularly interesting, then, is the fact if we regard (12) as a *renormalized* equation for (1), the renormalized equations give us a way to canonically extend the solution past any (geometric) singularity of the original time-like minimal submanifold problem.

Using the above ideas (particularly the fundamental solution), Nguyen and Tian [21] was able to show that for the compact string problem in ambient dimension $N = 2$, for arbitrary smooth initial data the solution collapses in finite time. They furthered conjectured that this phenomenon can be unstable in higher dimensions. This conjecture was verified by Jerrard, Novaga, and Orlandi [10], in which they showed that

(1) In ambient dimensions $N \geq 3$, the solutions can be divided into two non-empty classes: either a solution exists globally in *both* past and future time, or it collapses in finite time for *both* future and past evolutions.
(2) Furthermore, when $N = 3$, both classes have non-empty interior (and thus there are initial data of either class that are stable under small perturbations);
(3) While in ambient dimensions $N \geq 4$, global existence is generic (initial data exhibiting finite time blow-up can be perturbed with arbitrarily small perturbations to ones that exist globally in time).

To understand the dimensional dependence, let us quickly examine the proofs. A fundamental idea used in both works is that the failure for ϕ to remain an immersion is tied to certain degeneracies of the conformal structure given by the induced metric g. When ϕ is an immersion, the induced metric g is Lorentzian, and hence over M we can choose two continuous families of independent vector fields

u, v such that at every point $\phi_* u$ and $\phi_* v$ are null vectors in $\mathbb{R}^{1,N}$. The renormalized equation (12) implies that along the integral curves of u, the vectors $\phi_* v$ are parallel *as vectors in* $\mathbb{R}^{1,N}$, and vice versa. Furthermore, the mapping ϕ can be seen to fail as an immersion *precisely when, at a given point, $\phi_* u$ and $\phi_* v$ become parallel.*

Putting everything together, whether a given initial data set leads to finite time blow-up in the cosmic string case can be entirely determined by checking whether the vector fields $\phi_* u$ and $\phi_* v$, which can be defined at the initial data level, are such that for two points $y_1, y_2 \in \mathbb{S}^1$, $\phi_* u(y_1)$ is parallel to $\phi_* v(y_2)$. Now, the null directions in $\mathbb{R}^{1,N}$ can be parametrized by the celestial sphere, which is a copy of $\mathbb{S}^{N-1}$. And so the discussion finally reduces to the intersection properties of closed curves on $\mathbb{S}^{N-1}$, and from which we see the topological explanation of the stability when $N = 2, 3$ and the instability when $N \geq 4$ of collapsing solutions.

5. Singularities for axially-symmetric compact membranes

The speaker has recently obtained singularity formation results for some classes of higher dimensional time-like minimal submanifolds [26]. The results can be thought of as a hybrid generalizing both the result of Nguyen and Tian and the result in full spherical symmetry, both described in the previous section. The results further provide descriptions of the singular behavior of the solution, in the spirit of comparing the results against that driven by shock-formation mechanisms such as those described in section 3.

The first step of the analysis is to obtain a suitably renormalized equation similar to (12). The main tool used is that of the co-moving gauge. Recalling $t = x^0 \circ \phi$ as the induced time-function on M and Σ_t its level sets, we can also define the vector field τ along M by the conditions that $\tau(t) = 1$ and $\tau \perp_g \Sigma_t$. We will denote the norm $g(\tau, \tau) = -\alpha^2$, and we note that $\alpha \in (0, 1]$ by construction.

By $h = h(t)$ we denote the induced Riemannian metric on Σ_t. Of particular importance is the relativistic conservation of mass formula

$$\mathcal{L}_\tau \left(\alpha^{-1} \,\mathrm{dArea}_h \right) = 0 \tag{13}$$

which can be derived as a direct consequence of (1) for $A = 0$. This implies that the behavior, observed in the cosmic string case, where failure of ϕ to be an immersion is tied to degeneracy of the conformal structure of g, carries over in the same way: in particular, $\alpha \to 0$ (degeneracy of conformal structure) is tied to $\mathrm{dArea}_h \to 0$ (failure of ϕ to be an immersion). The renormalized equation can be found by noting

$$\alpha^2 \Box_g = -\tau^2 + \alpha h^{ij} \partial_i \alpha \partial_j + \alpha^2 \triangle_h. \tag{14}$$

Theorem 9 (Finite time singularity)**.** *Suppose there exists some $k \geq 2$ such that an isometric action of $SO(k)$ on $\mathbb{R}^{1,N}$ leaves M invariant with only a measure zero set of fixed points, then M blows up in finite time (could be either past or future).*

Sketch of proof. Let r be the radial coordinate on the subspace $\mathbb{R}^k$ on which $SO(k)$ acts. A computation shows that $\Box_g(r \circ \phi) \geq 0$. Plugging this into (14) and using (13), we can integrate over Σ_t to obtain the following *Virial inequality*

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \int_{\Sigma_t} \frac{r}{\alpha} \,\mathrm{dArea}_h \leq 0.$$

By convexity the integral must reach zero in finite time, which shows finite time collapse. $\square$

This theorem and its proof captures the essence behind the finite time collapse in the spherically symmetric case, and requires only the presence of one dimension that exhibits the symmetry. In order to give more precise description of the singularity mechanism, we have to impose a few more symmetries.

Definition 10. A solution M is said to be axially symmetric if there exists $d_1, \ldots, d_k$ such that $1 + \sum d_i = d = \dim(M) - 1$, such that

- M is parametrized by $(t, y, \omega_i|_{i=1,\ldots,k})$ where t takes value in an interval and $y \in \mathbb{S}^1$ while $\omega_i \in \mathbb{S}^{d_i}$;
- there exists mappings $z = z(t,y) \in \mathbb{R}^{N-d+1-k}$, $r_i|_{i=1,\ldots,k} = r_i(t,y) \in \mathbb{R}_+$, such that
$$\phi(t, y, \omega_1, \ldots, \omega_k) = (t, z, r_1\omega_1, \ldots, r_k\omega_k).$$

Remark 11. By Theorem 9, any axially symmetric solution must become singular in finite time.

Remark 12. As we will see below, the imposition of axial symmetry is technical in nature. It remains to be seen how much of the theorem below can be recovered after relaxing this symmetry condition.

Observe that while the manifold M has d spatial dimensions, the only real degree of freedom is in y. The equations of motion reduces to wave equations on a $1+1$ dimensional background. This reduction, coupled to the special form of our equation (such as the formulae (13) and (14)), allows some technical simplifications. However, as the geometry of the manifold M is really that in $1+d$ dimensions, tricks like "conformal invariance" cannot be used and the simplifications are not as striking. One of the key ideas in the analysis is to observe that we still have some gauge freedom in choosing how to parametrize $\mathbb{S}^1$ by the coordinate y. To wit, in the local coordinates given in the above definition, the equation of motion (14) can be shown to simplify to

$$-\partial^2_{tt} z + \frac{|g_{tt}|}{\sqrt{|\det g|}} \partial_y \left(\frac{\sqrt{|\det g|}}{|g_{yy}|} \partial_y z \right) = 0$$
$$-\partial^2_{tt} r_j + \frac{|g_{tt}|}{\sqrt{|\det g|}} \partial_y \left(\frac{\sqrt{|\det g|}}{|g_{yy}|} \partial_y r_j \right) = \frac{d_j |g_{tt}|}{r_j}$$

Using the one dimensionality of y together with (13), one can in fact find a reparametrization of y such that

$$\frac{|g_{tt}|}{\sqrt{|\det g|}}$$

is a constant on M. This implies that the equations of motion finally reduce to, for some constant $C > 0$,

$$-\partial^2_{tt} z + C \partial_y \left(r_1^{2d_1} r_2^{2d_2} \cdots r_k^{2d_k} \partial_y z \right) = 0 \tag{15a}$$

$$-\partial^2_{tt} r_j + C \partial_y \left(r_1^{2d_1} r_2^{2d_2} \cdots r_k^{2d_k} \partial_y r_j \right) = \frac{d_j |g_{tt}|}{r_j} \tag{15b}$$

and so we see that any solution to this system can be classically extended (by standard theory of nonlinear wave equations), provided that $\inf r_1 r_2 \cdots r_k > 0$ and that r, z are bounded in $W^{1,\infty}$.

Next, note however, that r, z, nor their time derivatives can blow-up in finite time, due to M being, throughout its time of existence, a time-like submanifold. Our gauge choice then further imposes that the spatial derivatives cannot blow-up either, save in the case when $(r_1 \cdots r_k) \searrow 0$. This shows that the finite-time blow-up for the *systems of equations* (15) can only be due to the radii product $r_1 \cdots r_k$ decreasing to zero. Recall, however, that the original geometric problem can still become singular when $\alpha \searrow 0$. This allows us to state:

Theorem 13 (Axial symmetry singular geometry)**.** *The finite time singularity of axially symmetric solutions is associated to the product $\alpha r_1 \cdots r_k \searrow 0$, and hence is due to loss of immersivity of the mapping ϕ. Furthermore, in the case $r_1 \cdots r_k$ remains bounded from below, the solution has a canonical weak extension past the singularity.*

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