Blow-up of QNLW with Small Initial Data
à la Christodoulou

A Geometric Perspective on Shock Formation

based on joint work with S. Klainerman and J. Speck

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Non-Linear wave equations

A few words about the structure of this mini-course.
We will be interested in understanding

$$-\partial_t^2 \phi + \Delta\phi + A^{\alpha\beta}(\phi, \partial\phi)\,\partial^2_{\alpha\beta}\phi = \mathcal{N}(\phi, \partial\phi)\ .$$

with small initial data $\phi(0, \mathbf{x}) = \epsilon f(\mathbf{x})$, $\partial_t \phi(0, \mathbf{x}) = \epsilon g(\mathbf{x})$.

When do solutions blow-up and when do we have small data global existence?

It will be useful to have the following four models in mind:

1. $-\partial_t^2\phi + \Delta\phi = (\partial_t\phi)^2 - |\nabla\phi|^2$ [Klainerman]

   null condition, small data global existence

2. $-\partial_t^2\phi + \Delta\phi = -(\partial_t\phi)^2$ [John]

   small data blow-up but no information on what goes wrong.

3. $-\partial_t^2\phi + (1+\phi)\,\Delta\phi = 0$ [Lindblad]

   small data global existence

4. $-\partial_t^2\phi + (1+\phi)\,\Delta\phi = -\frac{(\partial_t\phi)^2}{1+\phi}$ [John, Christodoulou]

   small data blow-up and information on (one particular form of) blow-up mechanism

We shall now investigate all four models **in spherical symmetry, in the wave zone**.

I. Spherical Symmetry

Linear waves in spherical symmetry

Consider the linear wave equation in spherical symmetry

$$\partial_v \partial_u (r\psi) = 0$$

where $t = u + v$ and $r = v - u$ and data of compact support in $[R_1, R_2]$. We will be particularly interested in the region near the wave front, i.e. the strip $\mathcal{R} := \{(u, v) \mid u_0 \leq u \leq u_1 \text{ and } v \geq v_0\}$.

wave front picture with $|r\psi| \leq C$, $|r\partial_u \psi| \leq C$ and $|r^2 \partial_v \psi| \leq C$

We solve $\partial_v \partial_u (r\psi) = 0$ “integrating along characteristics":

$$\partial_u (r\psi)(u, v) = \partial_u (r\psi)(u, v_{data}) \qquad \text{hence} \quad |\partial_u (r\psi)| \leq C$$

Integrating from the cone $u = u_0$ where the solution is zero, $r\psi(u, v) = 0 + \int_{u_0}^{u} du' \partial_{u'} (r\psi)$ yields (assuming bounded width)

$$|r\psi| \leq C \ .$$

Then from $r\partial_u \psi = \partial_u (r\psi) + \psi$ we have

$$|r\partial_u \psi| \leq C \, .$$

Finally, from $\partial_u (r\partial_v \psi) = \psi_u$ we find

$$|r^2 \partial_v \psi| \leq C \, .$$

In terms of decay, **the $\partial_u \psi$ derivative decays the slowest.**

A non-linear problem with null condition

Let us be more ambitious and try to "solve" a non-linear problem.

$$\begin{cases} \partial_v \partial_u (r\psi) = r \partial_u \psi \cdot \partial_v \psi \\ r\psi (t=0, r) = \epsilon f(r) \\ \partial_v (r\psi)(t=0,r) = \epsilon g(r) \end{cases} \tag{1}$$

This corresponds to the famous $\Box\psi = (\partial_t \psi)^2 - |\nabla \psi|^2$ with spherically symmetric data. [The problem can be solved with a simple algebraic trick but we don't want to use this.]

As this is a non-linear problem, we will need to do a bootstrap.

Let $\mathcal{B} \subset [v_0, \infty)$ be the subset consisting of those v' such that the following estimates hold in $\mathcal{R} \cap \{v < v'\}$:

$$|r\psi| \leq C\epsilon \quad , \quad |r\psi_u| \leq C\epsilon \quad , \quad |r^2 \psi_v| \leq C\epsilon$$

For sufficiently large C, the region $\mathcal{B}$ is non-empty and closed.

To show $\mathcal{B}$ is also open we show that the equation implies that the above estimates hold in $\mathcal{R} \cap \{v < v'\}$ with constant $\frac{C}{2}$.

To do this, we simply repeat the linear argument using the bootstrap assumptions:

$$|\partial_u (r\psi)| \leq \epsilon + C^2\epsilon^2 \int_{v_0}^{v} dv \frac{1}{r^2} \leq 2\epsilon$$

This implies

$$|r\psi| \leq 3\epsilon \quad \text{from integrating in the short direction}$$

$$|r\psi_u| \leq 5\epsilon \quad \text{from combining the previous bounds}$$

Finally, from $\partial_u (r\partial_v\psi) = -\psi_u + r\partial_u\psi\partial_v\psi$

$$|r\psi_v| \leq \frac{5\epsilon}{r} + C^2\epsilon^2\frac{1}{r^2} \leq \frac{C}{2}\epsilon \cdot \frac{1}{r}$$

for sufficiently small ϵ.

This establishes small data global existence in the strip. Clearly, what was crucial was that the right hand side was integrable. Indeed, for $r\left(\partial_u \psi\right)^2 \sim \frac{1}{r}$ this would not work.

A general result of F. John shows that $\Box\psi = \left(\partial_t \psi\right)^2$ does not admit global solutions for sufficiently small data but does not provide information on “what goes wrong".

A glimpse into a possible blow-up mechanism can be seen from

$$\partial_v \left(r\partial_u \psi\right) = \psi_v + \frac{1}{r}\left(r\partial_u \psi\right)^2 \ .$$

Ignoring the linear term and using that $r \sim v$ in the strip $\mathcal{R}$ we are lead to the ODE

$$\partial_s f = \frac{1}{s} f^2 \quad , \quad f\left(s_0\right) = F \quad \text{with solution } f\left(s\right) = \frac{1}{\frac{1}{F} - \log \frac{s}{s_0}}$$

Exploiting this mechanism requires controlling all other quantities and understanding the behavior of the solution until the blow-up (as well as ensuring that nothing goes wrong earlier), which is very difficult. For instance, note that

$$\partial_u \left(r\partial_v \psi \right) = \psi_u + \frac{1}{r} \left(r\partial_u \psi \right)^2 \ .$$

so it looks like that $r\partial_v\psi$ will also blow up.

It is remarkable that the next model we are going to study, a *quasi-linear* model, provides a setting to study this.

- only $r\partial_u\psi$ becomes singular
- $r\psi$ and $r^2\partial_v\psi$ remain small

Such a singularity can truly be called a shock.

Fritz John's model

Consider the following *quasi-linear* equation in spherical symmetry

$$\begin{cases} -\partial_t^2 \phi + (1 + \phi_t)\, \Delta\phi = 0 \\ \phi\,(t = 0, \vec{x}) = \epsilon f\,(r) \\ \phi_t\,(t = 0, \vec{x}) = \epsilon g\,(r) \end{cases} \tag{2}$$

It relates to the previous equation in that setting $\psi = \partial_t \phi$ we have

$$-\partial_t^2 \psi + (1 + \psi)\, \Delta\psi = -\frac{(\partial_t \psi)^2}{1 + \psi}$$

The quasi-linear part will in fact help us!

Theorem 1. *Fix f, g of compact support with $f^2 + g^2 \neq 0$. Then there exists an ϵ_0 such that for any $\epsilon < \epsilon_0$ the solution of*

$$\begin{cases} -\partial_t^2 \phi + (1 + \phi_t) \Delta \phi = 0 \\ \phi (t = 0, \vec{x}) = \epsilon f (r) \\ \phi_t (t = 0, \vec{x}) = \epsilon g (r) \end{cases} \tag{3}$$

must blow up in finite time. In fact, $T_{max} < \exp\left(C \frac{1}{\epsilon}\right)$.

The result will be proven by considering the solution near the wave-front, i.e. in the region $\mathcal{R}$. It could be that something bad happens (slightly) earlier in the interior!

In $\mathcal{R}$, we will actually show that ϕ_t and ϕ_r remain small all the way to the blow-up, while certain second derivatives of ϕ blow up.

In spherical symmetry we have

$$-\partial_t^2 (r\psi) + (1+\psi)\,\partial_r^2 (r\psi) = -r\frac{(\partial_t \psi)^2}{1+\psi}$$

and we can solve again using the method of characteristics.

The characteristic directions are

$$L = \partial_t + \sqrt{1+\psi}\partial_r \qquad \text{and} \qquad \underline{L} = \partial_t - \sqrt{1+\psi}\partial_r\,.$$

And we have two optical functions $u(t,r)$ and $\underline{u}(t,r)$ satisfying

$$Lu(t,r) = 0 \quad \text{and} \quad \underline{L}\,\underline{u}(t,r) = 0$$

so $u = const$ defines the outgoing characteristics C_u. We define the important quantity (foliation density)

$$\mu^{-1} = \partial_t u(t,r)$$

measuring the density of the leaves C_u.

Let us see what happens if we write the wave equation as a transport equation along characteristics.

$$L\underline{L} = \left(\partial_t + \sqrt{1+\psi}\partial_r\right)\left(\partial_t - \sqrt{1+\psi}\partial_r\right)$$
$$= \partial_t^2 - (1+\psi)\,\partial_r^2 - \frac{1}{2\sqrt{1+\psi}}\partial_t\psi\partial_r - \frac{1}{2}\partial_r\psi\partial_r$$

In particular,

$$L\underline{L}\,(r\psi) = r\frac{(\partial_t\psi)^2}{1+\psi} - \frac{1}{2\sqrt{1+\psi}}L\psi\partial_r\,(r\psi)$$
$$= \frac{1}{4}\frac{r}{1+\psi}\left(L\psi + \underline{L}\psi\right)^2 - \frac{1}{4}\frac{r}{1+\psi}L\psi\left(L\psi - \underline{L}\psi\right) + ...$$
$$= \frac{1}{4}\frac{r}{1+\psi}\left[\left(\underline{L}\psi\right)^2 + 3L\psi\underline{L}\psi\right] - \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi\cdot L\psi$$

We have

$$
\begin{aligned}
[L, \underline{L}](\psi r) &= -\frac{1}{\sqrt{1+\psi}}\psi_t \partial_r (\psi r) \\
&= -\frac{1}{4}\frac{1}{1+\psi}\left(L\psi + \underline{L}\psi\right)\left(L(\psi r) - \underline{L}(r\psi)\right) \\
&= -\frac{1}{4}\frac{r}{1+\psi}\left[(L\psi)^2 - (\underline{L}\psi)^2\right] - \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi\left(L\psi + \underline{L}\psi\right)
\end{aligned}
$$

so that interchanging L and $\underline{L}$ yields

$$
\begin{aligned}
L\underline{L}(r\psi) &= \frac{1}{4}\frac{r}{1+\psi}\left[(\underline{L}\psi)^2 + 3L\psi\underline{L}\psi\right] - \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi \cdot L\psi \\
\underline{L}L(r\psi) &= \frac{1}{4}\frac{r}{1+\psi}\left[(L\psi)^2 + 3L\psi\underline{L}\psi\right] + \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi \cdot \underline{L}\psi
\end{aligned}
$$

Dangerous quadratic term has disappeared from the second equation compared with the semi-linear case!

$$L\underline{L}\,(r\psi) = \frac{1}{4}\frac{r}{1+\psi}\left[(\underline{L}\psi)^2 + 3L\psi\underline{L}\psi\right] - \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi\cdot L\psi$$

$$\underline{L}L\,(r\psi) = \frac{1}{4}\frac{r}{1+\psi}\left[(L\psi)^2 + 3L\psi\underline{L}\psi\right] + \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\psi\cdot \underline{L}\psi$$

Actually we want

$$L\left(\frac{\underline{L}\,(r\psi)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left[(\underline{L}\psi)^2 - L\psi\underline{L}\psi\right] + lot$$

$$\underline{L}\left(\frac{L\,(r\psi)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left[(L\psi)^2 - L\psi\underline{L}\psi\right] + lot$$

Let us compute the derivative of the foliation density:

$$L\mu^{-1} = L\partial_t u\,(t,r) = \partial_t L u\,(t,r) - [\partial_t, L]\, u\,(t,r)$$

where we recall

$$Lu\,(t,r) = 0 \quad ,\text{and} \quad \underline{L}u\,(t,r) = (\underline{L} + L)\, u\,(t,r) = 2\partial_t u\,(t,r) = 2\mu^{-1}$$

and in view of

$$\begin{aligned}[\partial_t, L] &= \partial_t \left(\partial_t + \sqrt{1+\psi}\partial_r\right) - \left(\partial_t + \sqrt{1+\psi}\partial_r\right)\partial_t \\ &= \frac{1}{2}\frac{1}{\sqrt{1+\psi}}\partial_t\psi\partial_r\end{aligned}$$

we find

$$L\mu^{-1} = \frac{1}{4}\frac{1}{1+\psi}\left(L\psi + \underline{L}\psi\right)\left(\mu^{-1}\right)$$

and hence

$$L\mu = -\frac{1}{4}\frac{1}{1+\psi}\mu\left(L\psi + \underline{L}\psi\right)\,.$$

This allows us to remove the blow-up term

$$L\left(\mu\frac{\underline{L}\,(r\psi)}{1+\psi}\right)=\frac{1}{4}\frac{r}{(1+\psi)^2}\left[-2L\psi\cdot\mu\underline{L}\psi\right]+l.o.t.$$

$$\mu\underline{L}\left(\frac{L\,(r\psi)}{1+\psi}\right)=\frac{1}{4}\frac{r}{(1+\psi)^2}\left[\mu\,(L\psi)^2-L\psi\cdot\mu\underline{L}\psi\right]+l.o.t.$$

The claim is that we should be able to bootstrap

$$|r\psi|\leq C\epsilon\quad,\quad|\mu r\underline{L}\psi|\leq C\epsilon\quad,\quad|r^2L\psi|\leq C\epsilon$$

in the region $\mathcal{R}\cap\{t\leq T\}$ for T the time of existence. At the same time, however,

$$\mu\to 0\quad\text{and}\quad|r\underline{L}\psi|\to\infty$$

Note that in the (u,t) coordinate system $\boxed{L=\partial_t}$ and $\boxed{\mu\underline{L}=\partial_u}$.

To refresh your memory:

From (t,r) we defined a new coordinate system $\left(u,\widetilde{t}\right)$ by

$$u = u\,(t,r) \qquad \text{and} \qquad \widetilde{t} = t$$

with $u\,(t,r)$ satisfying the eikonal equation, i.e. the transformation depends on the solution itself! Since

$$0 = Lu\,(t,r) = \partial_t u\,(t,r) + \sqrt{1+\psi}\,\partial_r u\,(t,r) \quad \text{and} \quad \partial_t u\,(t,r) = \mu^{-1}$$

we find

$$\partial_t = \frac{1}{\mu}\partial_u + \partial_{\widetilde{t}} \quad \text{and} \quad \partial_r = -\frac{1}{\sqrt{1+\psi}}\partial_u + 0\cdot\partial_{\widetilde{t}}$$

hence

$$L = \partial_{\widetilde{t}} \qquad \text{and} \qquad \underline{L} = \mu^{-1}\partial_u$$

Let's prove that indeed for as long as the solution of the system

$$L\left(\mu\frac{\underline{L}\left(r\psi\right)}{1+\psi}\right)=\frac{1}{4}\frac{r}{\left(1+\psi\right)^{2}}\left[-L\psi\cdot\mu\underline{L}\psi\right]+l.o.t.$$

$$\mu\underline{L}\left(\frac{L\left(r\psi\right)}{1+\psi}\right)=\frac{1}{4}\frac{r}{\left(1+\psi\right)^{2}}\left[\mu\left(L\psi\right)^{2}-L\psi\cdot\mu\underline{L}\psi\right]+l.o.t.$$

exists, we have

$$|r\psi|\leq C\epsilon \quad , \quad |\mu r\underline{L}\psi|\leq C\epsilon \quad , \quad |r^{2}L\psi|\leq C\epsilon$$

in the region $\mathcal{R}\cap\{t\leq T\}$ for T the time of existence.

In fact, this is very simple. Do it on the board (assuming also an *upper* bound on μ). Note also that in view of

$$L\left(t-r-u\left(t,r\right)\right)=1-\sqrt{1+\psi}=\mathcal{O}\left(\epsilon\cdot r^{-1}\right)$$

the leaves C_u diverge logarithmically from the Minkowskian ones.

This was Step 1: Certain "renormalized" quantities remain small for as long as the solution exists! This is the global existence part.

We still need to prove bounds on μ (an upper bound for the global existence and the "blow-up" estimate $\mu \to 0$ promised earlier).

Recall our previous formula

$$L\mu = -\frac{1}{4}\frac{1}{1+\psi}\mu\,(L\psi + \underline{L}\psi)$$

It follows that μ can grow at most logarithmically, which is sufficient to close the "global existence" estimates above.

For the lower bound, recall that

$$L\left(\mu\frac{\underline{L}\left(r\psi\right)}{1+\psi}\right)=\frac{1}{4}\frac{r}{\left(1+\psi\right)^{2}}\left[-2L\psi\cdot\mu\underline{L}\psi\right]+l.o.t.$$

Suppose $\mu\underline{L}\left(r\psi\right)=+c\cdot\epsilon$ holds at one point on the data. Then it follows that along the corresponding C_u

$$\mu\underline{L}\left(r\psi\right)>\frac{c}{2}\cdot\epsilon$$

and therefore, after waiting sufficiently long ($t\sim r\sim\epsilon^{-\frac{1}{2}}$ say)

$$r\cdot\mu\cdot\underline{L}\left(\psi\right)>\frac{c}{4}\cdot\epsilon$$

holds along C_u and hence

$$L\mu\leq-\frac{c}{8}\cdot\epsilon\cdot\frac{1}{r}$$

After time $t \sim \epsilon^{-\frac{1}{2}}$ the quantity μ is still close to 1 (just integrate $L\mu$ to get $1 - C\epsilon \log \epsilon^{-\frac{1}{2}}$). Integrating

$$L\mu \leq -\frac{c}{8} \cdot \epsilon \cdot \frac{1}{r}$$

along C_u yields

$$\mu - \mu(s_0) \leq -\frac{c}{8} \cdot \epsilon \cdot \log \frac{s}{s_0}$$

which shows that μ has to blow up after a time of size $\exp(1/\epsilon)$.

Finally, the fact that $\underline{L}(r\psi)$ must be positive at some point on the data can be deduced from the assumption of compact support. (Directly seen via characteristic data.)

Final Comments

- Relating it back to John's problem is straightforward
- More general non-linearities can be implemented (we will obtain a systematic understanding later!)
- Comparison with John's original proof
- Comparison with Lindblad's example (next slide)

Consider Lindblad's model (weak null condition)

$$\begin{cases} -\partial_t^2\psi + (1+\psi)\,\Delta\psi & = 0 \\ \psi\,(t=0,\vec{x}) & = \epsilon f\,(r) \\ \psi_t\,(t=0,\vec{x}) & = \epsilon g\,(r) \end{cases} \tag{4}$$

There is no term driving blow-up on the right hand side in this case.

So our old set of equations

$$L\left(\frac{\underline{L}\,(r\psi)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left[(\underline{L}\psi)^2 - L\psi\underline{L}\psi\right] + lot$$

$$\mu\underline{L}\left(\frac{L\,(r\psi)}{(1+\psi)}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\mu\left[(L\psi)^2 - L\psi\underline{L}\psi\right] + lot$$

changes to

$$L\left(\frac{\underline{L}\,(r\psi)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left[(L\psi)^2 + L\psi\underline{L}\psi\right] + lot$$

$$\mu\underline{L}\left(\frac{L\,(r\psi)}{(1+\psi)}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\mu\left[(\underline{L}\psi)^2 + L\psi\underline{L}\psi\right] + lot$$

The right hand side of the first is integrable (with lots of room).
The second doesn't lose r when integrating.

If we believe that μ can grow a bit, say

$$|\mu| + \frac{1}{|\mu|} \leq 2t^{\frac{1}{4}} \quad \text{for } t \geq 1,$$

Then we can bootstrap

$$|rL\psi| \leq C \cdot \epsilon \cdot \frac{1}{t^{\frac{3}{4}}} \quad , \quad |r\underline{L}\psi| \leq C \cdot \epsilon \quad , \quad |r\psi| \leq C \cdot \epsilon$$

At the same time we still have

$$L\mu = -\frac{1}{4}\frac{1}{1+\psi}\mu\left(L\psi + \underline{L}\psi\right)$$

which gives

$$\mu \leq t^{C\epsilon} \quad \text{and} \quad \frac{1}{\mu} \leq t^{C\epsilon}$$

So in this case, *global existence* with the cones deviating from their Minkowskian cones but **no blow-up in finite time**.

Away from spherical symmetry...

Keep in mind the main structure of the proof:

1. Introduction of a renormalized frame (via eikonal equation) $\Longrightarrow$ global existence with respect to that frame.
2. Blow-up of μ^{-1} via transport equation

This structure will be maintained. However, away from spherical symmetry we cannot use pointwise estimates to close! There are extremely subtle regularity issues, coupled with the μ-weights.

It is known that away from spherical symmetry only L^2 estimates do not lose regularity for the wave equation. Part II of our course will develop these techniques in a geometric context.

II. Vector Field Method and Radiative Decay

Shock formation, in our context, can be described as similar to a "coordinate singularity". (Alinhac calls this "geometric blow-up".)

The classic example is that of Burger's equation

$$u_t + uu_x = 0$$

which we can resolve using the method of characteristics. The solution remains constant on any characteristic curve, and shock formation is due to *intersecting* characteristics.

The characteristic curves can be described by the map $x = X(t, y)$ where y is the initial position $y = X(0, y)$. Another way of saying that we have a shock is that the solution u is *regular* in the coordinate system (t, y) (it is constant along each fixed y); the singularity is due to the coordinate transformation $(t, y) \mapsto (t, X(t, y))$ being non-invertible (more precisely the Jacobian determinant goes to zero).

The same splitting can also be seen in the example described in the first lecture:

The weight by μ, the inverse foliation density, cancels out precisely the zero in the denominator given by the Jacobian determinant when the coordinate system degenerates.

In the μ-weighted frame, the *solution to the wave equation* remains perfectly regular.

The singularity manifests only when converting back to a “physical” frame, which involves division by μ (again, the Jacobian in the change of variables), which goes to 0 at the singular point.

We aim to prove shock formation for certain quasilinear wave equations of the geometric form

$$\Box_{g(\psi)}\psi = 0$$

where $\Box_g$ is the Laplace-Beltrami operator (or coordinate-invariant wave operator) associated to a Lorentzian metric g.

The "blow up" term is hidden in the Christoffel symbols. In the "geometric" picture we are again trying to prove "global existence". (Lindblad's example now has a bad right hand side in this picture, which exactly kills the blow up term.)

Emulating the process described in the first lecture, we need to

- find a rescaled frame, and prove in the rescaled frame that ψ is regular,
- derive evolution equations for the frame (in particular, for μ or the Jacobian determinant), and show that singularity forms.

In this second lecture we will review the vector field method, which will be used to drive the part of the analysis that shows the regularity of ψ.

In part III we will derive the evolution equations for the frame.

And in the final lecture we will show how to estimate the various quantities that comes up and tie everything together.

L^2 energy estimates

- Spherically symmetric case: integrated along characteristics. Gives direct L^∞ control. Can do this because equation reduces to $1+1$.
- In the non symmetric case, more degress of freedom. Can only use L^2 estimates when derivative loss can be problematic (nonlinear problems).
- Example:
  In spherical coordinates the standard wave operator is

  $$\Box = (-\partial_t + \partial_r)(\partial_t + \partial_r) + \frac{d-1}{2r}[(-\partial_t + \partial_r) + (\partial_t + \partial_r)] + \frac{1}{r^2}\not{\Delta}$$

  treating as pure transport will lose angular derivatives!

Energy estimates for the linear wave equation

- Restrict to first discussion the linear wave equation $\Box\psi = 0$ for clarity. Recall
$$\Box = -\partial_t^2 + \sum \partial_{x^i}^2$$
is the usual wave operator.

- Restrict the analysis to just the "wave zone". (Though vector field method can also be used more generally.) This is because that's where shock happens.

- Suppose our data is supported in the ball of radius 1.

- Define the wave zone: for $\epsilon, T > 0$,
$$\mathcal{W}_{\epsilon,T} := \{t \in (0,T), r - t \in (1-\epsilon, 1)\}.$$
Note that by finite speed of propagation the solution vanishes identically on the exterior region $r - t > 1$.

- Define $u = t - r + 1$, so the wave zone $\mathcal{W}_{\epsilon,T}$ is expressed equivalently as $\{t \in (0,T), u \in (0,\epsilon)\}$.
- Observe that u is *null* relative to the Minkowski metric:

$$-(\partial_t u)^2 + \sum (\partial_{x^i} u)^2 = 0 \ .$$

- The level subsets corresponding to t and u we write as

$$\Sigma_{\epsilon,\tau} := \{t = \tau, u \in (0,\epsilon)\},$$
$$\mathcal{C}_{u_0,T} := \{t \in (0,T), u = u_0\}.$$

- $\Sigma_{\epsilon,\tau}$ are annuli and $\mathcal{C}_{u_0,T}$ are truncated cones.

Stress-energy tensor and conservation law

Stress-energy tensor:

$$T_{\mu\nu}^{(\phi)} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m_{\mu\nu}m^{\sigma\tau}\partial_\sigma\phi\partial_\tau\phi$$

where $m = \mathrm{diag}(-1,1,1,1)$ is the Minkowski metric. The superscript just clarifies the function involved, and will often be suppressed in what follows.

Direct computation $\implies m^{\lambda\mu}\partial_\lambda T_{\mu\nu} = \Box\phi\partial_\nu\phi$ so if X^ν is any vector field, we have the conservation law

$$\begin{aligned}\partial_\lambda J^\lambda_{(X,\phi)} &= \partial_\lambda\left(m^{\lambda\mu}T_{\mu\nu}X^\nu\right)\\ &= \Box\phi X(\phi) + T_{\mu\nu}m^{\lambda\mu}(\partial_\lambda X^\nu) = I_{(X,\phi)}.\end{aligned}$$

Like all good analysts, we integrate total divergences.

$$\int_\Omega (\mathrm{div} X) \mathrm{dvol} = \int_{\partial\Omega} \iota_X \mathrm{dvol}$$

The volume form in the coordinates (u, t, ω) where $\omega \in \mathbb{S}^{d-1}$ can be written schematically as $r^{d-1}\mathrm{d}u\mathrm{d}t\mathrm{d}\omega$. Remark that the boundaries of the wave zone $\mathcal{W}_{\epsilon,T}$ are level sets of t, u. We have

$$\begin{aligned}\int_0^T \int_0^\epsilon \int_{\mathbb{S}^{d-1}} I\ r^{d-1}\mathrm{d}\omega\mathrm{d}u\mathrm{d}t &= \int_0^\epsilon \int_{\mathbb{S}^{d-1}} J(t)\ r^{d-1}\mathrm{d}\omega\mathrm{d}u\Big|_{t=0}^{t=T} \\ &\quad + \int_0^T \int_{\mathbb{S}^{d-1}} J(u)\ r^{d-1}\mathrm{d}\omega\mathrm{d}t\Big|_{u=0}^{u=\epsilon} .\end{aligned}$$

This is the fundamental *energy identity*.

If instead of an arbitrary function ϕ, we look at a solution to $\Box\psi = 0$ with data supported inside the unit ball, we note that in the identity

$$\int_0^T \int_0^\epsilon \int_{\mathbb{S}^{d-1}} I \; r^{d-1} \mathrm{d}\omega \mathrm{d}u \mathrm{d}t = \int_0^\epsilon \int_{\mathbb{S}^{d-1}} J(t) \; r^{d-1} \mathrm{d}\omega \mathrm{d}u \Big|_{t=0}^{t=T} + \int_0^T \int_{\mathbb{S}^{d-1}} J(u) \; r^{d-1} \mathrm{d}\omega \mathrm{d}t \Big|_{u=0}^{u=\epsilon} ,$$

- I simplifies to just $m^{\lambda\mu} T_{\mu\nu} \partial_\lambda X^\nu$.
- The boundary integral along $u = 0$ vanishes, since $T_{\mu\nu} \equiv 0$ there.

Multiplier vector fields; deformation tensor

The usefulness of the energy identity depends on the vector field X. Ideally to get control of quantities we need coercivity. A sufficient condition for $J(t)$ and $J(u)$ to be signed is for X to be causal relative to the Minkowski metric.

Among causal vector fields, we choose vector fields for which the factor $\partial_\lambda X^\nu$ which appears in I either vanishes, or leads to I with good properties. For convenience we write

$$^{(X)}\pi^{\mu\nu} = m^{\lambda\mu}\partial_\lambda X^\nu$$

and call it the *deformation tensor* of the vector field X.

Example: $X^\nu = \partial_t$

$$J^\lambda_{(X,\phi)} = -\frac{1}{2}\left[(\partial_t\phi)^2 + \sum(\partial_{x^i}\phi)^2\right]\partial_t + \sum(\partial_t\phi\partial_{x^i}\phi)\partial_{x^i}$$

and in particular

$$J(t) = -\frac{1}{2}\left[(\partial_t\phi)^2 + |\nabla\phi|^2\right] \le 0$$
$$J(u) = -\frac{1}{2}(\partial_t\phi + \partial_r\phi)^2 - \frac{1}{2}|\nabla\!\!\!/\phi|^2 \le 0$$

On the other hand $\partial_\lambda X^\nu = 0$ and so for solutions ψ we have $I_{(X,\psi)} = 0$. This leads to the energy identity

$$\int_{\mathcal{C}_{\epsilon,T}} (\partial_t\psi + \partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 \text{ dA} + \int_{\Sigma_{\epsilon,T}} (\partial_t\psi)^2 + |\nabla\psi|^2 \text{ dA}$$
$$= \int_{\Sigma_{\epsilon,0}} (\partial_t\psi)^2 + |\nabla\psi|^2 \text{ dA}$$

A first decay estimate

The positivity of the integral over $\mathcal{C}_{\epsilon,T}$ allows us to rewrite as

$$\int_{\Sigma_{\epsilon,T}} (\partial_t\psi)^2 + |\nabla\psi|^2 \text{ dA} \leq \int_{\Sigma_{\epsilon,0}} (\partial_t\psi)^2 + |\nabla\psi|^2 \text{ dA}$$

Using that the area element $\text{dA} = r^{d-1}\text{d}\omega\text{d}r \approx t^{d-1}\text{d}\omega\text{d}r$ in the wave zone $\mathcal{W}_{\epsilon,T}$, this is in fact a *decay estimate*.

Assuming the sizes of $\partial\psi$ remains roughly homogeneous on the annuli $\Sigma_{\epsilon,\tau}$, the energy inequality would imply that

$$|\partial_t\psi(t,\cdot)|, |\nabla\psi(t,\cdot)| \lesssim (1+t)^{\frac{1-d}{2}} \cdot \text{initial data}$$

which is compatible with the spherical symmetry result.

To actually go from L^2 estimates to L^∞ ones, we use Sobolev embedding theorem. This requires taking higher derivatives, which we address later.

Cheap 0th-order control

The energy identity above controls only the derivatives $\partial\psi$. Assuming that we can implement the L^∞ estimate, for the derivaties, recovering an L^∞ level control for ψ itself is just a matter of integrating: here we use that the wave zone has bounded thickness!

$$|\psi(t,\cdot)| \leq \int_0^\epsilon |\partial_r \psi(t)| \, \mathrm{d}u$$

Example: $X^\nu = r^2(\partial_t + \partial_r)$

A computation yields

$$J(t) = -\frac{r^2}{2}\left[(\partial_t\phi + \partial_r\phi)^2 + |\nabla\!\!\!/\phi|^2\right]$$
$$J(u) = -r^2(\partial_t\phi + \partial_r\phi)^2$$

with $\partial_t X^\nu = 0$, $\partial_i X^0 = 2x^i$, and $\partial_i X^j = x^i x^j/r + r\delta_i^j$. So

$$I = \underbrace{r(\partial_t\phi + \partial_r\phi)^2}_{\geq 0} - \frac{d-1}{2} r[-(\partial_t\phi)^2 + |\nabla\phi|^2].$$

The second component of I we treat using the method of "modified currents" (i.e. systematically integrating by parts), noting that for ψ solving $\Box\psi = 0$:

$$\partial_\mu(m^{\mu\nu}\psi\partial_\nu\psi) = -(\partial_t\psi)^2 + |\nabla\psi|^2.$$

This leads eventually to the energy identity

$$\textit{Flux} + \textit{Top Energy} + \textit{Good Bulk} = \textit{Initial Energy}$$

where we have the positive terms

$$\textit{Flux} = \int_{\mathcal{C}_{\epsilon,T}} 2\left(r\partial_t\psi + r\partial_r\psi + \frac{d-1}{2}\psi\right)^2 \text{ dA}$$

$$\_\_\_\_ \ \textit{Energy} = \int_{\Sigma_{\epsilon,\tau}} \left(r\partial_t\psi + r\partial_r\psi + \frac{d-1}{2}\psi\right)^2 + \frac{(d-1)(d-3)}{4}\psi^2 + r^2|\nabla\!\!\!/\psi|^2 \text{ dA}$$

$$\textit{Good Bulk} = \int_{W_{\epsilon,T}} 2r(\partial_t\psi + \partial_r\psi + \frac{d-1}{2r}\psi)^2 \text{ dvol}$$

Improved tangential decay estimates

Let us assume, for the time being, that the solution is roughly homogeneous over the annulus. The cheap zeroth order control implies that the L^2 integral

$$\int_{\Sigma_{\epsilon,\tau}} \psi^2 \mathrm{dA} \lesssim Data.$$

Combining with the the energy estimate from the previous slide

$$Top\ Energy \leq Data$$

we obtain

$$\int_{\Sigma_{\epsilon,T}} r^2(\partial_t\psi + \partial_r\psi)^2 + r^2|\nabla\!\!\!/\psi|^2 \,\mathrm{dA} \lesssim Data$$

On the left hand side, we estimate $r \approx t$ and this leads to

$$\int_{\Sigma_{\epsilon,T}} (\partial_t \psi + \partial_r \psi)^2 + |\nabla\!\!\!/ \psi|^2 \text{ dA} \lesssim \frac{1}{T^2} \cdot \mathit{Data}$$

Compare this to the first decay estimate:

- Reduced control: d (space-time) directions instead of $d + 1$.
- Improved decay: extra T^2 decay in energy, or extra T decay pointwise.

Note that the missing direction is precisely the one transverse to level sets of u; *cannot* improve decay of ψ itself.

Higher order energy and commutators

There are two reasons to control higher order energies

1. As mentioned above, we use Sobolev inequality to translate L^2 control (with derivatives) to L^∞ control

2. In applications, we also need L^∞ control of higher derivatives.

To do so we write down equations satisfied by "suitable derivatives" of ψ. Let Z be a vector field, then we have trivially, for a solution ψ of the wave equation:

$$\Box(Z\psi) = [\Box, Z]\psi$$

so if $[\Box, Z]$ is controllable (say, $\equiv 0$), then the same technique above can be used.

The highest order error term introduced is ${}^{(Z)}\pi^{\mu\nu}\partial^2_{\mu\nu}(\psi)$
$\Longrightarrow$ (approximate) symmetries of the equation comes into play.

For Sobolev

We are integrating over the annulus, and the r^{d-1} factor in the volume form is used for the decay. Pulling out this factor we are essentially working on slices of the thickened cylinder $(0,\epsilon)\times(0,T)\times\mathrm{S}^{d-1}$. So for uniform Sobolev control we need to use derivatives adapted to this cylinder.

The vector fields should then be

- A radial derivative
- Angular derivatives

Lifting back to the original space-time, they can be controlled by

- Spatial gradients ∂_{x^i}
- Rotational vector fields $\Omega_{ij} = x^i\partial_j - x^j\partial_i$

Both sets commute with $\Box$.

Improved higher-order tangential decay

As already seen in the spherically symmetric case, the derivatives *transverse* to level sets of u do not decay faster than the rate given by the first energy estimate. But tangential derivatives behave better.

To capture this we need to higher order energy estimates for weighted tangential vector fields. $(d-1)$ spatial directions are furnished by the rotations Ω_{ij}. For the remaining direction (basically $\partial_t + \partial_r$), we study $Z = r\partial_t + r\partial_r$.

Note: this is connected with the classical vector field method in which we commute with the scaling vector field $t\partial_t + r\partial_r$, which is a conformal symmetry of the wave equation:

$$[\Box, t\partial_t + r\partial_r] = 2\Box$$

So for solutions

$$\Box(t\partial_t\psi + r\partial_r\psi) = 0.$$

Now, we can decompose

$$r(\partial_t + \partial_r) = t\partial_t + r\partial_r + \partial_t - u\partial_t$$

The first three terms commute well with $\Box$. The remaining term can be computed explicitly

$$[\Box, u\partial_t] = (\Box u)\partial_t + 2m^{\mu\nu}\partial_\mu u \partial^2_{\nu t} = -\frac{d-1}{r}\partial_t - 2(\partial_t + \partial_r)\partial_t$$

In other words, if ψ solves $\Box\psi = 0$ then

$$\Box(r(\partial_t + \partial_r)\psi) = \frac{d-1}{r}\partial_t\psi + 2(\partial_t + \partial_r)\partial_t\psi.$$

For the energy estimate this has favorable structure: its contribution to the first energy estimate, is adding to I the terms

$$(d-1)(\partial_t + \partial_r)(\partial_t\psi)^2 + 2r[(\partial_t + \partial_r)\partial_t\psi]^2$$

the first is boundary + good sign and the second has good sign.

Example application: semilinear wave equation in 3+1

Let us return now to the equation on $\mathbb{R}^{1+3}$

$$\Box\psi = m^{\mu\nu}\partial_\mu\psi\partial_\nu\psi$$

treated in the first lecture. In this equation there is an inhomogeneous term, which contributes extra to the energy integral. To prove global existence it suffices to show that this extra contribution is suitably small.

We can do so, again, using bootstrap assumptions. From our energy estimates we expect in the wave zone

$$|r\nabla\!\!\!/\psi|, |r(\partial_t + \partial_r)\psi| \lesssim (1+t)^{-1} Data$$

In the first energy estimate, for example, the contribution to I given by the inhomogeneity can be decomposed as

$$-(\partial_t\psi + \partial_r\psi)(\partial_t\psi - \partial_r\psi)\partial_t\psi + \nabla\!\!\!/\psi\nabla\!\!\!/\psi\partial_t\psi$$

In each term we take the first factor in L^∞ and estimate the remaining two factors by the energy, we have that its contribution, when integrated, is bounded by

$$\int_0^T \frac{1}{(1+\tau)^2}\int_{\Sigma_{\epsilon,\tau}} |\partial_t\psi|^2 + |\nabla\psi|^2 \mathrm{dA}\ \mathrm{d}\tau \cdot Data$$

Using that $(1+\tau)^{-2}$ is integrable, the "damage" caused by the nonlinearity can be easily absorbed (using Gronwall, for example) in the energy estimate provided that $Data$ is sufficiently small.

Caveats

In the quasilinear case, there are some obvious changes to what I wrote above.

1. The choice of the multiplier vector fields will need to depend on the quasilinear metric $g(\psi)$. This leads to
   - More complicated boundary integrals for the energies; in particular weights in μ! In the shock case this poses some difficulty with coercivity etc. (Lecture IV)
   - More complicated bulk terms coming from error terms due to the deformation tensor of the multiplier vector fields.
2. Similarly, the choice of the commutator vector fields will also need to be adapted to the geometry. This will also introduce additional error terms coming from the commutator against the wave operator.

The error terms, since they come from $g(\psi)$, we expect to be effectively *nonlinear* contributions to the bulk term I. Therefore our choice of the commutator and multiplier vector fields are motivated by requiring that the error terms satisfy something like the null condition:

In fact we need to only have error terms that contain at least one factor of $(\partial_t \psi + \partial_r \psi)$ or $\nabla\!\!\!/\psi$ or something decaying at similar strength so that they carry an integrable decaying weight. Some aspects of this choice of vector fields will be discussed in the next lecture.

III. Null Geometry

One of the main innovations given by Christodoulou in demonstrating shock formation is the close attention paid to the geometry of the *quasilinear metric* $g(\psi)$. In particular,

1. the space-time domain of integration in the energy estimate and its foliation,

2. the multiplier vector fields,

3. and the commutator vector fields, and

4. the energy quantities controlled

are all adapted to the geometry. In this session we will introduce the null geometry and indicate the first three items in the list. Item 4 will be discussed as part of the final lecture.

Optical function

In the previous lecture we introduced the function $u = t - r + 1$ on Minkowski space, and we remarked that it is *null* relative to the Minkowski metric. This function played an important role in the discussion: the directions in which derivatives for a solution of the wave equation have improved decay are precisely tangent to the level sets of u.

So the first step to understanding the null geometry is to understand its replacement over a curved background.

The fundamental property that makes u useful is that it is *null*, or, in other words, solves the *eikonal equation*

$$g^{\mu\nu}(\psi)\partial_\mu u \partial_\nu u = 0 \ .$$

This makes its level sets characteristic surfaces for the wave operator $\Box_{g(\psi)}$. (We often refer to u as the *optical function*.)

The eikonal equation is invariant up to reparametrisation (chain rule).

We chose u such that $u = 1 - r$ on the initial data slice.

The wave zone and its foliation

That the level sets of u are null implies that energy fluxes (the boundary integrals in the energy identity) are signed (as long as the multiplier fields are causal). This makes them well suited to be boundary components of the wave zone.

The wave zone in the quasilinear case can be defined analogously to that in the linear case:

$$\mathcal{W}_{\epsilon,T} := \{t \in (0,T), u \in (0,\epsilon)\}$$

where t is the Minkowskian time coordinate, and u the optical function as above.

We can also foliate $\mathcal{W}_{\epsilon,T}$ using the level sets of u (the sets $\mathcal{C}_{u_0,T}$) and t (the sets $\Sigma_{\epsilon,\tau}$) exactly as in the linear case.

Let also $S_{u_0,\tau} = C_{u_0,T} \cap \Sigma_{\epsilon,\tau}$ be the spheres of intersection.

Reminder of intuition

Shock formation is the crossing of characteristics. This means that we expect to see onset of shock formation reflected in the mapping $(t, u, \omega) \to \vec{x} = (x^0, x^1, x^2, x^3)$ being singular: that $\partial_u \vec{x} = 0$ in the change of variables.

In terms of the level sets, this is suggesting that level surfaces of u becomes infinitely dense at some point. Before making precise how we measure this density we have to introduce one more definition.

The $\widehat{L}$ vector field

Let $\widehat{L}$ be the vector field whose components are defined by

$$\widehat{L}^\mu = -g^{\mu\nu}\partial_\nu u.$$

(The minus sign just to make it future directed.)

- $\widehat{L}(u) = -g^{\mu\nu}\partial_\nu u \partial_\mu u = 0$: $\widehat{L}$ is tangent to $\mathcal{C}_{u_0,T}$.
- $g_{\mu\nu}\widehat{L}^\mu \widehat{L}^\nu = g_{\mu\nu}g^{\mu\tau}\partial_\tau u g^{\nu\sigma}\partial_\sigma u = 0$: $\widehat{L}$ is null.
- $D_{\widehat{L}}\widehat{L} = -g^{\mu\nu}D_\nu[\widehat{L}(u)] = 0$: $\widehat{L}$ is geodesic. (D is the Levi-Civita connection associated to the Lorentzian metric g.)

This geodesic equation will be the means by which we develop transport equations for geometric quantities!

Inverse foliation density μ

The component $\widehat{L}^0 = \widehat{L}(x^0)$, where $x^0 \equiv t$ is the Minkowski time coordinate, can be written also as

$$-g^{\mu\nu}\partial_\nu t \partial_\mu u$$

Shock formation $\implies$ g regular (ψ cannot itself blow-up)
$\implies$ $g^{\mu\nu}\partial_\nu t$ is regular vector field, *transverse* to u.
$\implies$ this derivative should blow up if u's level sets are infinitely dense.

So we define

$$\mu := \frac{1}{\widehat{L}^0} \geq 0$$

to be the *inverse foliation density*. Its sign comes from $\widehat{L}$ being future directed.

In John's equation, the inverse quasilinear metric is

$$g^{-1}(\phi_t) = \begin{pmatrix} -1 & & & \\ & 1+\phi_t & & \\ & & 1+\phi_t & \\ & & & 1+\phi_t \end{pmatrix}$$

and hence the μ defined here agrees with that defined in the spherically symmetric example given in the first lecture:

$$\partial_t u = -g^{\mu\nu}\partial_\mu t \partial_\nu u \ .$$

(In general this equation is false.)

We then define the *time-normalised* null vector

$$L = \mu \widehat{L}.$$

Observe that

- While the components of $\widehat{L}$ blows up as shock forms, the components of L remains bounded.
- In fact, $L(x^0) = 1$.
- Just like $\widehat{L}$, L is null, tangent to $\mathcal{C}_{u_0,T}$, and future-directed.
- L is not geodesic: instead it solves

$$D_L L = D_L(\mu \widehat{L}) = \frac{L(\mu)}{\mu} L$$

A normalisation choice

We will assume that our quasilinear metric g satisfies

$$g^{\mu\nu}\partial_\mu t \partial_\nu t = -1$$

where t is the Minkowskian time function.

One can always fix one single component of the metric via a conformal rescaling. While this introduces inhomogeneities into the wave equation, it can be shown (a simple algebraic computation) that the additional terms all satisfy the null condition, and is harmless in terms of the full analysis.

$\Longrightarrow$ this choice is made without loss of generality. It howevers simplifies computations.

Adapted frame

We define $\breve{R}$ to be the unique vector field satisfying

- Tangent to Σ: $\breve{R}(t) = 0$
- Normalised to u: $\breve{R}(u) = 1$
- Sphere orthogonal: $\breve{R}$ is g-orthogonal to the spheres $S_{u_0,\tau}$.

It is our "∂_u". By definition it is transverse to the u foliation, and so the second condition implies that when u becomes infinitely dense, $\breve{R}$ *degenerates*.

Its "regular" counterpart is obtained by dividing by μ: we write

$$R = \mu^{-1}\breve{R}.$$

A direct computation using the *normalisation condition* of the previous slide shows that $g(R, R) = 1$ everywhere.

The normalisation condition also implies that $L + 2R$ is a null vector (note that R is *inward directed* from our choice of u!)

Hence we define $\underline{L} := L + 2R$. We can complete the frame by choosing an orthonormal frame on $S_{u,\tau}$ relative to the induced Riemannian metric, we write them as X_1, X_2.

We will interchangeably use the *null frame* $\{L, \underline{L}, X_1, X_2\}$ as well as the frame $\{L, R, X_1, X_2\}$.

Analogous to $\breve{R}$ sometimes we will also write

$$\breve{\underline{L}} = \mu \underline{L}.$$

With or without μ weights

- $\breve{R}$ and $\breve{\underline{L}}$ should be used as *differential operators*: they are our ∂_u and derivatives relative to them are expected to remain regular.

- When decomposing differential operators (vector fields), use these guys. Recall in our analysis of the John equation, to obtain "global existence" we looked at the transports

$$L\left(\frac{\mu\underline{L}(r\psi)}{1+\psi}\right) = \dots \qquad \text{and} \qquad \mu\underline{L}\left(\frac{L(r\psi)}{1+\psi}\right) = \dots$$

- R and $\underline{L}$ should be used for *frame decompositions* when we try to estimate components of a tensor object. They are size-1 objects that do not degenerate.

- $R\psi$, however, is the "physical derivative" that should blow up when shock forms.

Let's recall what we are doing. . .

In the spherically symmetric case we studied the coupled system of equations for ψ and u . . .

. . . and u can be completely controlled by μ: the metric gives a simple relation between $\partial_t u$ and $\partial_r u$.

When there is angular dependence, we also need to control angular derivatives of u; such terms would also come up when rewriting the wave equation "in the (u, t) coordinates" (more precisely expressing the wave equation in terms of the frame).

Controlling u is exactly the same as controlling the geometry of the foliations C_u, and this in term can be captured in the Ricci rotation coefficients of the adapted frame!

Historical remark

A very similar shock formation result was also obtained by S. Alinhac (through a series of work in the last 90s ending in two papers around 2000)

- The basic geometric motivation is the same
- Very different techniques:
  - work directly at the level of the optical function u instead of the frames
  - uses a Nash-Moser type argument to "perturb from Burger"...
  - ...so can use cheaper (easier to prove) estimates with derivative loss

The many derivatives of u

- μ and L controls $\widehat{L}$ and hence first derivatives of u.
- Control μ and L by **geodesic equation** for $\widehat{L}$.
- Commute equation to get higher order control.
- Higher derivative control of u also given by rotation coefficients

$$g(D_{e_1}L, e_2)$$

where e_1, e_2 are frame vectors. They appear naturally in the wave equation too.

- For example, the null second fundamental form

$$\chi_{AB} = g(D_{X_A}L, X_B) = \mu g(D_{X_A}\widehat{L}, X_B)$$

is the covariant Hessian of u in the purely angular directions.

- The evolution of Ricci coefficients governed by **null structure equations**.

Geodesic equation

Relative to Minkowski coordinates, we have

$$(D_L L)^\mu = L^\nu \partial_\nu L^\mu + \Gamma^\mu_{LL}$$

where the Christoffel symbol

$$\Gamma^\mu_{LL} = \frac{1}{2} g^{\mu\sigma} \left(2G_{L\sigma} L(\psi) - G_{LL} \partial_\sigma \psi\right)$$

and where

$$G_{\mu\nu}(\psi) = \frac{\mathrm{d}}{\mathrm{d}\psi} g_{\mu\nu}(\psi).$$

That $L^0 = 1$ gives from the geodesic equation and the normalisation choice

$$L(\mu) = \frac{1}{2} \mu G_{LL} R(\psi) + O(\mu L(\psi)).$$

The L^i (spatial) components obey better equations. Plugging in the expression for $L(\mu)$ we get

$$L(L^i) = \Gamma^0_{LL} L^i - \Gamma^i_{LL} L^0$$

the antisymmetrisation precise gets rid of the worst term $R(\psi)$ in the Christoffel symbol, and we have that

$$L(L^i) = O(L(\psi), X_A(\psi))$$

And of course $L(L^0) = 0$ trivially.

We see that these equations give

$$L(\mu, L^i) = \text{terms depending on first derivatives of } \psi$$

If we integrate the transport, we have that u has the same regularity as ψ in terms of estimates.

Null structure equations

- Well studied in the context of general relativity
- Roughly speaking: connects two definitions of curvature
  - derivatives of the Ricci coefficients (3rd derivatives of u)
  - second derivatives of the metric (second derivatives of ψ)
- Another way of thinking about them: coordinate derivatives of u commute $\implies$ integrability conditions.

Decomposition of the wave equation

The principal part of the wave operator $\Box_g$ is $g^{\mu\nu}\partial^2_{\mu\nu}$; g, however, expands in $\{L, \underline{L}, X_A\}$ with size 1 coefficients.

Since $\Box_g$ is differential operator, with the intuition above we should consider instead $\mu\Box_g$ which is decomposed naturally in $\{L, \breve{\underline{L}}, X_A\}$ with size 1 coefficients:

$$\mu\Box_{g(\psi)}\psi = L\breve{\underline{L}}\psi + \mathrm{tr}\chi\breve{\underline{L}}\psi + \mathrm{tr}\not{k}L\psi - \mu\not{\Delta}\psi + \mathrm{junk}$$

This in turn forces us to study μ-weighted energy estimates (next section).

Here, however, we see a potential problem with the regularity if we proceed naively with energy estimates: recall that χ has regularity of the Hessian of u. So the direct argument for energy estimates will control n-derivatives of ψ using $n+1$ derivatives of u...

…but as we've seen above, integrating the transport equations gives us that the regularity of u is the same of ψ

$\implies$ we face a potential loss of derivatives.

Of course, this is not surprising! This is the same as studying a quasilinear wave equation by considering the quasilinearity purely as a source term instead of incorporating it in the energy quantity.

To close the argument one needs to find a way to "fix" this apparent loss which is essentially due to expressing the equations in a non-optimal (regularity-wise) way.

Following Christodoulou, we implement the fix at the level of the transport equations. (Will be discussed next time.)

Some hints to the resolution can be seen in spherical symmetry: in spherical symmetry, the regularity of χ is better! Recall that in spherical symmetry χ is pure trace, and its trace is the null expansion, and is expressed as the null derivative of the area of the symmetry spheres.

For the spherically symmetric metric $g(\psi)$ this area is a function of ψ, t, and u

$\implies \chi$ in spherical symmetry is on the order of 1 derivative of u and of ψ.

One expects then the bad behaviour is somehow connected to angular derivatives; and that is precisely what we will see used in the next lecture.

Unweighted wave equation

Above we have looked at the μ-weighted wave equation based on our intuition that "good" differential operators (for which we can prove "global existence") uses $\breve{R}$ instead of R. What happens if we look at the bad equation?

Leaving out the μ weight from the wave equation, and absorbing the χ factor by proper r weights, we have that

$$L((1-u+t)R\psi) = (1-u+t)(R\psi)^2 + \not{\Delta}\psi + junk$$

Provided we control the angular derivative terms the blow-up mechanism is exactly the same as the case in spherical symmetry.

Therefore, once we closed the "long time existence" part, the blow-up is relatively simple; in particular it only needs to be carried out "at the lowest level of derivatives".

Multiplier fields

Recall that in the linear wave equation we used the two multiplier fields: ∂_t and $r^2(\partial_t + \partial_r)$.

The geometric analogue of the second is clear: replace r by the geometric $(t - u + 1)$ and the $\partial_t + \partial_r$ by the outgoing null vector field L and we have that Morawetz vector field

$$K_{(1)} = (1 - u + t)^2 L$$

For the first term, we have roughly that $\partial_t = L + R$. But as the multiplier field acts as a derivative of ψ in the energy inequality, our intuition above tells us that it is better to look at $\mu(L + R) = \mu L + \breve{R}$ to get regularity. This vector field, however, completely degenerates when $\mu = 0$. So it turns out that to recover some control on the energy in this limit we correct it a bit further and use

$$K_{(0)} = (1 + 2\mu)L + 2\breve{R}.$$

Commutator fields

The choice of the commutator fields Z is to minimize the damage caused by the commutators $[\mu\Box, Z]$ when writing down the equation satisfied by $\mu\Box Z(\psi)$.

Possible dangers in the inhomogeneity:

1. Loss of regularity
2. Loss of decay
3. Loss of μ weights

Regularity loss is not possible from commutation: commutators of differential operators are necessarily lower order, and the energy estimate relative to $K_{(0)}$ is coercive in all directions.

In terms of decay loss, the main enemy is for $\breve{R})(\breve{R}\psi)$ terms to appear in the commuted equations. This would be the cases when the deformation tensor

$$^{(Z)}\pi^{\mu\nu} = \nabla^\mu Z^\nu + \nabla^\nu Z^\mu$$

has non-zero coefficient in front of $R^\mu R^\nu$ in frame decomposition.

For μ loss, we need to make sure the error terms do not contain additional factors of μ^{-1}.

To recover all derivatives we also need the commutator fields to span the tangent space. Emulating the linear case we will use

- Weighted derivatives tangent to the C_u foliation
- Unweighted derivative transversal to the C_u foliation.

We will use

- Approximate scaling $(1+t-u)L$
- Projected angular momentum Ω_{ij}
- $\breve{R}$

By our assumption $\breve{R}(u) = 1$: for any C_u tangential vector field Y,

$$Y\breve{R}(u) - \breve{R}Y(u) = 0 \implies [Y, \breve{R}] \parallel C_u.$$

Now since the principal part of $\mu\Box_g$ decomposes as

$$\mu LL + 2L\breve{R} - \mu\not{\Delta}$$

the top order commutator against our vector fields above must have at least one factor that is tangent to C_u, avoiding decay loss.

Lack of $\breve{R}$ terms in the vector field commutators also indicates that μ loss is unlikely: can verify by computation.

IV. Shock Formation for Quasilinear Geometric Wave Equations

Recall that we introduced the frame $\left(L, \breve{\underline{L}}, X_1, X_2\right)$.

The fundamental equations are

$$L\breve{\underline{L}}\psi + tr\chi\breve{\underline{L}}\psi + tr\not{k}L\psi = \mu\not{\Delta}\psi + \text{junk}$$

and the transport equation for the foliation density

$$L\mu = \frac{1}{2}G_{LL}\left(\breve{R}\psi\right) + \text{harmless}$$

The multiplier vectorfield $K_{(0)} = (1 + 2\mu)\, L + \breve{R}$ will generate the energy

$$E^2_{K_{(0)}}[\psi](t) = \int_{\Sigma_t} d\underline{\varpi} \left[\psi^2 + \mu\, (L\psi)^2 + \left(\breve{R}\psi \right)^2 + \mu |\nabla\!\!\!/ \psi|^2 \right]$$

and the flux

$$E^2_{K_{(0)}}[\psi](u) = \int_{t_0}^{t} d\overline{\varpi} \left[(L\psi)^2 + \mu |\nabla\!\!\!/ \psi|^2 \right]$$

Note the various degenerations. In particular, the transversal derivative is only controlled on Σ_t (in a non-degenerate fashion).

→ Explain heuristically how this comes about.

The multiplier vectorfield $K_{(1)} = (1-u+t)^2 L$ will generate the energy

$$E^2_{K_{(1)}}[\psi](t) = \int_{\Sigma_t} d\underline{\varpi} \left[\mu \left(L (1-u+t) \psi \right)^2 + \mu (1-u+t)^2 |\nabla\!\!\!/ \psi|^2 \right]$$

and the flux

$$E^2_{K_{(1)}}[\psi](u) = \int_{t_0}^{t} d\overline{\varpi} \left[\left(L (1-u+t) \psi \right)^2 \right]$$

Note the various degenerations and t-weights. This is in accord with our view that "L-derivatives are harmless."

→ Explain heuristically how this comes about.

The energy estimates

Recall from Section 2+3 that commuting the wave equation with a vectorfield $\mathcal{Z}$ yields

$$\Box_{g(\psi)}\left(\mathcal{Z}^k[\psi]\right)=\mathcal{C}\left[\mathcal{Z}^k\psi\right]$$

and (reducing to $k=1$) we have the *energy identity*

$$\begin{aligned}E^2_{K_i}[\mathcal{Z}\psi](t)+I^2_{K_i}[\mathcal{Z}\psi](\mathcal{M}_{t,u})+F^2_{K_i}[\mathcal{Z}\psi](u)\\=E^2_{K_i}[\mathcal{Z}\psi](t_0)+\int_{\mathcal{M}_{t,u}}\mathcal{C}\left[\mathcal{Z}^k\psi\right]\cdot K_i\mathcal{Z}\psi\end{aligned}$$

- recall how this “closes" in the semi-linear null-condition case
- here coupling of μ-weights and t-weights will play crucial role

A good (non-linear!) space-time Morawetz term

While $I^2_{K_1}[\psi](\mathcal{M}_{t,u})$ is cubic and easily controlled, we find that

$$I^2_{K_1}[\psi](\mathcal{M}_{t,u}) = \int_{\mathcal{M}_{t,u}} (1-u+t)^2 (-L\mu) |\nabla\!\!\!/ \psi|^2 + \text{errors}$$

The point is that as $\mu \to 0$ we expect

$$-L\mu \geq +\frac{c}{r}\epsilon$$

recalling the blow-up mechanism (and also the radial case!).

This gives strong (without μ-degeneration!) control over the angular derivatives in the region where μ is small.

Let us now turn to the error-term arising from the commutator:

$$\int_{\mathcal{M}_{t,u}} \mathcal{C}\left[\mathcal{Z}^k \psi\right] \cdot K_i \mathcal{Z}\psi$$

This is the biggest difficulty. Let us understand the structure of these terms and how they can be controlled by the left hand side.

We commute by $(1-u+t)\,L$, $\breve{R}$ and the Ω_i. All of them satisfy

$$^{(\mathcal{Z})}\pi_{LL} = 0$$

This is important as it avoids dangerous $\breve{R}\breve{R}\psi$-terms in $\mathcal{C}\left[\mathcal{Z}^k\psi\right]$.

The L-commutation is always good but the others introduce potentially dangerous terms.

From the decomposed wave equation

$$L\breve{\underline{L}}\psi + tr\chi\breve{\underline{L}}\psi + tr\not{k}L\psi = \mu\not{\Delta}\psi + \text{junk}$$

we see that commuting by $\breve{R}$ introduces a term

$$\mu\Box_{g(\psi)}\left(\breve{R}\psi\right) = \breve{R}\psi\not{\Delta}\mu$$

and that commuting with Ω_i introduces a term

$$\mu\Box_{g(\psi)}\left(\Omega_i\psi\right) = -\breve{R}\psi\Omega_i\left(tr\chi\right)$$

We should think of the right hand side as **three** derivatives of the optical function.

Let us focus on the first term

$$\mu\Box_{g(\psi)}\left(\breve{R}\psi\right)=\breve{R}\psi\not\Delta\mu+\text{better}$$

and

$$L\left(\not\Delta\mu\right)=\frac{1}{2}G_{LL}\left(\breve{R}\not\Delta\psi\right)+\text{better}$$

The idea is to renormalize:

$$L\left(\not\Delta\mu\right)=\frac{1}{2}G_{LL}\left(\breve{R}\frac{1}{\mu}L\breve{R}\psi\right)+\text{better}$$

so

$$L\left(\not\Delta\mu-\frac{1}{2}\frac{1}{\mu}G_{LL}\breve{R}\breve{R}\psi\right)=\text{second derivatives of }\psi$$

But this introduces lower order terms degenerating badly in μ, so do it for $L\left(\mu\not\Delta\mu\right)=....$

In any case, morally

$$\not\Delta\mu = \frac{1}{2}\frac{1}{\mu}G_{LL}\breve{R}\breve{R}\psi + \text{terms under control}$$

so that our critical error-term in the $K_{(0)}$-estimate becomes

$$\int_{\mathcal{M}_{t,u}} \breve{R}\psi \not\Delta\mu \left(K_{(0)}\breve{R}\psi\right)$$
$$\sim \int_{\mathcal{M}_{t,u}} \breve{R}\psi\frac{1}{2}\frac{1}{\mu}G_{LL}\left(\breve{R}\breve{R}\psi\right)^2 \sim \int_{\mathcal{M}_{t,u}} \frac{L\mu}{\mu}\left(\breve{R}\breve{R}\psi\right)^2 .$$

We've gained a derivative but lost a power of μ!

In particular, when we Gronwall the energy identity, we find that the energy grows like

$$E^2_{K_{(0)}}\left[\breve{R}\psi\right](t) = E^2_{K_{(0)}}\left[\breve{R}\psi\right](t_0)\cdot\mu^{-C}$$

This closes at the top level but with very bad degeneration in μ. [One novel feature of our work is keeping track of the constant C.]

How do we descend to lower order non-degenerate energies? The idea is that at the next to top order, we can do something different to the error-term.

Suppose the top order was **two** commutations and that we did the above renormalization. We now turn to the energy estimate for the **once** commuted equation. Before doing so, we note

$$\int_{\Sigma_t} |\not\Delta\mu|^2 \leq E^2_{K_i}\left[\mathcal{Z}^2\right](t_0) \cdot \mu^{-C}$$

as follows from integrating the transport equation without renormalization.

The error-term of the 1-commuted equation is

$$
\begin{aligned}
&\int_{\mathcal{M}_{t,u}} \breve{R}\psi \not\Delta\mu \left(K_{(0)}\breve{R}\psi\right) \\
\lesssim\ &\int dt \left[\sup_{\Sigma_t}(-L\mu)\right] \int_{\Sigma_t} \left(|\not\Delta\mu|^2 + |K_{(0)}\breve{R}\psi|^2\right) \\
\lesssim\ &\int dt \left[\sup_{\Sigma_t}(-L\mu)\right] \mu^{-C} \\
\lesssim\ &\int dt \left[\frac{\epsilon}{1-u+t}\right] \frac{1}{\left(1-\epsilon\log(1-u+t)\right)^C} \\
\lesssim\ &\mu^{-C+1}\,.
\end{aligned}
$$

Actually, everything has to be done at the level of
$\mu_\star = \min\left(\inf_{\Sigma_t}\mu, 1\right)$.

This allows to descend and obtain non-degenerate estimates at the lowest level which then control the geometry.

Note that unlike in the spherically symmetric case here you need to bootstrap also the blow-up bound for μ (or rather $\mu_\star$) as it is crucial for the above descend scheme. It is retrieved from the transport equation for μ, just like in the spherically symmetric case.

We also need to impose a sign condition on the data (just as in the spherically symmetric case). Recall

$$L\left(\mu\frac{\underline{L}\,(r\psi)}{1+\psi}\right) = \frac{1}{4}\frac{r}{(1+\psi)^2}\left[-2L\psi\cdot\mu\underline{L}\psi\right] + \mu\cdot r\cdot \not{\Delta}$$

Suppose $\mu\underline{L}\,(r\psi) = +c\cdot\epsilon$ holds at one point on the data.
Now there is a linear (angular term)! To "preserve" the sign there are two possibilities

- impose sign condition on the spherical average (Christodoulou) **"Sign Condition A"**
- impose that angular derivatives are "smaller" ($\epsilon^{1+\delta}$ on the data). Then wait $t\sim\epsilon^{-\delta}$ when $\mu r\not{\Delta}\sim\frac{1}{r^2}\epsilon^{1+\delta}\epsilon^{-\delta}\sim\frac{1}{r^2}$ after which r is sufficiently large to provide smallness.
  "Sign Condition B"

Remarks about sign condition:

- Motivated by "preserving sign condition from initial data"
- One can alternatively allow data to possibly fail the sign condition initially but observe sign condition "asymptotically": impose that the *linear radiation field* of the initial data has a sign condition (Alinhac) **"Sign condition C"**
  - Radiation field $\Longrightarrow$ integral transform data
  - In spherical symmetry **A, B** and **C** are equivalent
  - Outside spherical symmetry **C** is expected to be the sharpest (**A** and **B** both implies **C** from decay estimates)
- In spherical symmetry it is easy to verify the sign conditions are verified automatically: essentially argue by mean value theorem and the fact we have compact support.
- Conditions **A, B** are stable under *very* small angular perturbations.

Obviously, there are many difficulties that we have swept under the rug. Some of them are:

1. The renormalization procedure for the $\Omega_i tr\chi$-error-term.
2. lower order terms (!)
3. elliptic estimates

Let us conclude by stating the actual results and put them them into context with Christodoulou's book.

Definition 1. *A non-linear geometric wave equation (for possible vector valued ψ)*

$$\Box_{g(\psi)}\psi = \mathcal{N}(\psi, \partial\psi)$$

is called Riccati-type shock forming if the following conditions hold:

1. *Violation of the null-condition: There exists at least one $\omega \in \mathbb{S}^2 \subset \mathbb{R}^3$ such that for the* Minkowskian *null vector $\ell = (1, \omega_1, \omega_2, \omega_3)$, our metric $g_{\mu\nu}(\psi)$ satisfies*

$$N(\omega) := G_{\mu\nu}(0)\ell^\mu \ell^\nu \neq 0\ .$$

2. *Right hand side satisfies strong null-conditon: The nonlinearity $\mathcal{N}$ can be decomposed as*

$$\mathcal{N}(\psi, \partial\psi) = \sum f_i(\psi) Q_i(\partial\psi, \partial\psi)$$

where the f_i's are smooth functions and Q_i's are null forms relative to the metric $g(\psi)$, i.e. $(Q_i)^{\alpha\beta} L_\alpha L_\beta = 0$ holds for co-vectors L with $g^{\alpha\beta} L_\alpha L_\beta = 0$.

Note that the quadratic blow-up term is hiding in $\Box_{g(\psi)}$:

$$\Box_{g(\psi)}\psi = \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}g^{ab}(\psi)\,\partial_b\psi\right) = g^{ab}(\psi)\,\partial_a\partial_b\psi + \frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}g^{ab}(\psi)\right)\partial_b\psi$$

In particular, we can revisit our favourite example (commuted John's model)

$$-\partial_t^2\psi + (1+\psi)\,\Delta\psi = -\frac{(\partial_t\psi)^2}{1+\psi}$$

We can write it as

$$\Box_{g(\psi)}\psi = -\frac{1}{2}\frac{(\partial_t\psi)^2}{1+v} + \frac{1}{2}|\nabla\psi|^2$$

for the metric $g = diag\left(-1, (1+v)^{-1}, (1+v)^{-1}, (1+v)^{-1}\right)$.

The right hand side satisfies the null-condition w.r.t. $g(\psi)$.

Theorem 2. *Consider*

- *a* **Riccati-type shock forming equation** *with*
- *initial data of compact support satisfying one of* **sign conditions A**, **B**, *or* **C** *(depend in part on angular structure of equation)*

Then there is are constants C_1, C_2 *such that,* $\mu_\star = min\left(\inf_{\Sigma_t} \mu, 1\right)$ *satisfies the estimates*

$$1 - \mu_\star \leq C_1 \epsilon \log\left(2 - u + t\right)$$
$$\mu_\star \leq 1 - C_2 \epsilon \log\left(2 - u + t\right)$$

Moreover, in the region where the solution exists,

$$|r\mu_\star R\psi| \leq C\epsilon \quad and \quad |r^{2-\delta} L\psi| + |r^{2-\delta} \nabla\!\!\!/ \psi| \leq C\epsilon\,.$$