LECTURE NOTES ON SOBOLEV SPACES
FOR CAMBRIDGE CENTRE FOR ANALYSIS

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0.1. References.

Before we start, some references:

- D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Springer. Ch. 7.
- L. Evans, *Partial differential equations*, American Math. Soc. Ch. 5.
- M. E. Taylor, *Partial differential equations I*, Springer. Ch. 4. (*Note:* this presentation is based on heavy doses of Fourier analysis and functional analysis.)
- H. Triebel, *Theory of function spaces*, Birkhauser. Ch. 2.
- R. Adams, *Sobolev Spaces*, Academic Press.

0.2. Notations.

We will work in $\mathbb{R}^d$.

p': given $p \geq 1$ a real number, we define p' to be the positive real number satisfying $p^{-1} + (p')^{-1} = 1$; p' is called the *Hölder conjugate* of p

Ω: open set in $\mathbb{R}^d$

$\partial\Omega$: the boundary of Ω, $\bar{\Omega} \setminus \Omega$

∂^α: partial derivative of *multi-index* α. $\alpha = (\alpha_1, \ldots, \alpha_d) \in (\mathbb{N}_0)^d$, with norm $|\alpha| = \sum \alpha_i$. $\partial^\alpha = \partial_1^{\alpha_1} \cdot \partial_d^{\alpha_d}$

$\Omega_1 \Subset \Omega_2$: there exists a compact set K such that $\Omega_1 \subset K \subset \Omega_2$

D^α: weak derivative (see §1.3) of multi-index α

$\operatorname{supp} f$: for a function f, this denotes the support set, i.e. the set on which $f \neq 0$

$C(\Omega)$: continuous functions taking value in the reals defined on Ω (though most of what we say will be valid for functions taking value in a Hilbert space)

$C(\bar{\Omega})$: the subset of $C(\Omega)$ consisting of functions that extend continuously to $\partial\Omega$

$C_0(\Omega)$: the subset of $C(\bar{\Omega})$ consisting of functions which vanish on $\partial\Omega$

$C^k(\Omega)$: functions f such that $\partial^\alpha f \in C(\Omega)$ for every $|\alpha| \leq k$. k is allowed to be ∞ (in which case f is smooth) or ω (in which case f is analytic). Analogously we define $C^k(\bar{\Omega})$ and $C_0^k(\Omega)$ (note that the set $C_0^\omega(\Omega)$ contains only the zero functions)

$C_c^k(\Omega)$: subset of $C^k(\Omega)$ such that $\operatorname{supp} f \Subset \Omega$

$L^p(\Omega), L^p_{\mathbf{loc}}(\Omega)$: Lebesgue spaces (see §1.1)

$W^{k,p}(\Omega), W^{k,p}_{\mathbf{loc}}(\Omega), W^{s,p}_0(\Omega)$: Sobolev spaces (see §1.4)

$\|\cdot\|_p$: L^p norm (see §1.1)

$\|\cdot\|_{p,k}$: $W^{k,p}$ norm (see §1.4)

Version `rev243` of `2013-07-10 08:06:40 -0400 (Wed, 10 Jul 2013)`.

1. Basic Definitions

In this first part Ω can be taken to be any open subset of $\mathbb{R}^d$. Throughout $\mathrm{d}x$ will be the standard Lebesgue measure. By a measurable function we'll mean a representative of an equivalence class of measurable functions which differ on Ω in a set of measure 0. Thus sup and inf should be mentally replaced by ess sup and ess inf when appropriate.

1.1. **Lebesgue spaces.**

For $\infty > p \geq 1$, $L^p(\Omega)$ denotes the set of p-integrable measurable functions, with norm

$$\|u\|_{p;\Omega} = \left(\int_\Omega |u|^p \, \mathrm{d}x \right)^{1/p} . \tag{1}$$

> If u takes values in some normed linear space, then $|\cdot|$ will be the Banach space norm.

For $p = \infty$, $L^\infty(\Omega)$ denotes the essentially bounded functions

$$\|u\|_{\infty;\Omega} = \sup_\Omega |u| . \tag{2}$$

For $1 \leq p \leq \infty$, the spaces $L^p(\Omega)$ are Banach spaces. The space $L^2(\Omega)$ is a Hilbert space, with inner-product

$$\langle u, v\rangle = \langle u, v\rangle_{0;\Omega} = \int_\Omega uv \, \mathrm{d}x . \tag{3}$$

> If u, v take values in a Hilbert space $\mathcal{H}$ with norm $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, then the integrand should be replaced by $\int_\Omega \langle u, v\rangle_{\mathcal{H}} \, \mathrm{d}(x)$. In particular, if they are complex valued, the integrand should be $\bar{u}v$.

For the sake of completeness, let me recall the definitions of Banach and Hilbert spaces. Let V be a linear space over $\mathbb{R}$.

> With the obvious substitutions, you can also do over $\mathbb{C}$

A norm $|\cdot|$ on V assigns to elements of V nonnegative real numbers, such that for $v, w \in V$:

(1) $|v| \geq 0$, with equality iff $v = 0$;
(2) $|sv| = |s|\,|v|$, for any scalar $s \in \mathbb{R}$;
(3) $|v + w| \leq |v| + |w|$ (triangle ineq.)

The pair $(V, |\cdot|)$ is called a *normed linear space*. We can define a metric on $(V, |\cdot|)$ by $d(v, w) = |v - w|$.

Exercise 1. Check that $d(v, w)$ is indeed a metric.

Then we can use the usual notion of convergence in metric spaces. If the normed linear space $(V, |\cdot|)$ is complete as a metric space, that is, all Cauchy sequences converge, then we say $(V, |\cdot|)$ is a *Banach space*. That the L^p norms are in fact norms follows easily from property of the Euclidean absolute value, and Hölder's inequality (6) below.

Exercise 2. Prove that $L^p(\Omega)$ is a Banach space. That is, show that if $u_i \in L^p(\Omega)$ are a sequence of functions satisfying $\|u_i - u_j\|_{p;\Omega} \to 0$ as $i, j \to \infty$, then there exists $u \in L^p(\Omega)$ such that $u_i \to u$.

Now let V be an $\mathbb{R}$-linear space again. An *inner product* on V is a map $V \times V \to \mathbb{R}$ denoted by $\langle \cdot, \cdot \rangle$, satisfying

(1) $\langle v, w\rangle = \langle w, v\rangle$;
(2) $\langle av + bw, x\rangle = a\langle v, x\rangle + b\langle w, x\rangle$ for all $a, b \in \mathbb{R}$;
(3) $\langle v, v\rangle \geq 0$, with equality iff $v = 0$.

We call the pair $(V, \langle \cdot, \cdot \rangle)$ an *inner product space* or *pre-Hilbert space*.

> If V were over $\mathbb{C}$, the symmetric property needs to be replaced by Hermitian, and the linearity needs to become sesquilinearity.

Exercise 3. Writing $\|v\| = \sqrt{\langle v, v\rangle}$ on an inner product space $(V, \langle \cdot, \cdot \rangle)$, show that

- $|\langle v, w\rangle| \leq \|v\| \, \|w\|$ (Schwarz)
- $\|v + w\| \leq \|v\| + \|w\|$ (Triangle)
- $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$ (Parallelogram identity)

This implies that *every inner product space is a normed linear space*.

If the induced norm is complete, we say an inner product space is a *Hilbert space*.

> In some texts a Hilbert space is also required to be separable: that there exists a countable subset of vectors $\{v_1, v_2, \ldots\}$ whose linear space is dense in V; this will in particular imply the Hilbert space has a countable orthonormal basis. While the spaces we encounter in this note are all, in fact, separable, we will not make this assumption.

Exercise 4. Check that $(L^2(\Omega), \langle \cdot, \cdot \rangle_{0;\Omega})$ is an inner product space. Check that the inner product defined in (3) indeed leads to the norm defined in (1). Therefore by Exercise 2, L^2 is a Hilbert space.

Recall *Young's inequality*: let a, b be positive real numbers, and $p \geq 1$ real, then

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'} \leq a^p + b^{p'} . \tag{4}$$

The special case $p = p' = 2$ is known as Cauchy's inequality. By replacing $a \to \epsilon a$ and $b \to \epsilon^{-1} b$, we also get the interpolated version of Young's inequality

$$ab \leq \epsilon^p a^p + \epsilon^{-p'} b^{p'} . \tag{5}$$

Using Young's inequality we can prove Hölder's inequality

$$\int_\Omega uv \, \mathrm{d}x \leq \|u\|_{p;\Omega} \|v\|_{p';\Omega} \tag{6}$$

when the right hand side is well defined.

Proof. The inequality is homogeneous in scaling of u and v. Therefore it suffices to prove for $\|u\|_p = \|v\|_{p'} = 1$. Write

$$\int_\Omega uv \, \mathrm{d}x \leq \int_\Omega |u|\,|v| \, \mathrm{d}x \leq \int_\Omega \frac{|u|^p}{p} + \frac{|v|^{p'}}{p'} \, \mathrm{d}x$$
$$\leq \frac{1}{p}\|u\|^p_{p;\Omega} + \frac{1}{p'}\|v\|^{p'}_{p';\Omega} = \frac{1}{p} + \frac{1}{p'} = 1$$

□

> This homogeneous scaling trick is *very useful* in analysis of partial differential equations.

Hölder's inequality is very useful for deriving facts about the L^p spaces. First we define the *localised Lebesgue spaces*: a function u is said to be in $L^p_{\mathrm{loc}}(\Omega)$ if for every open set $\Omega' \Subset \Omega$, $u \in L^p(\Omega')$.

> Consider the functions on $\mathbb{R}$ given by $|x|^{-\alpha}$. These functions are not in any Lebesgue spaces, because for $p\alpha \leq 1$, it decays too slowly at infinity, while for $p\alpha \geq 1$, it blows up too fast at the origin. The localised spaces allows one to distinguish divergences at the boundary of Ω, and singularities in the interior of Ω. Also note that the local Lebesgue spaces are not normed spaces.

Proposition 1. *(1) $L^q(\Omega) \subset L^p(\Omega)$ if $q \geq p$ and $|\Omega| < \infty$; (2) $L^p(\Omega) \cap L^q(\Omega) \subset L^r(\Omega)$ if $p \leq r \leq q$.*

Note that point (1) in particular implies that $L^q_{\mathrm{loc}}(\Omega) \subset L^p_{\mathrm{loc}}(\Omega)$ for $p < q$ and any Ω.

Proof. What we will prove are slightly stronger, quantitative versions of the above statements. For the first claim, let χ_Ω be the characteristic function of Ω, i.e. $\chi_\Omega(x) = 1$ if $x \in \Omega$ and 0 otherwise. Let $u \in L^q(\Omega)$. Observe

$$\begin{aligned}
\|u\|^p_{p;\Omega} &= \int_\Omega |u|^p \, \mathrm{d}x = \int_\Omega \chi_\Omega |u|^p \, \mathrm{d}x \\
\text{(by Hölder)} \quad &\leq \|\chi_\Omega\|_{1/(1-\frac{p}{q});\Omega} \|u^p\|_{q/p;\Omega} \\
&= |\Omega|^{1-\frac{p}{q}} \|u\|^p_{q;\Omega}
\end{aligned}$$

and so we have

$$\|u\|_{p;\Omega} \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|u\|_{q;\Omega} \tag{7}$$

which proves the claim. For the second statement, we write

$$\|u\|^r_{r;\Omega} = \int_\Omega |u|^{\lambda r} |u|^{(1-\lambda)r} \, \mathrm{d}x = \|u^{\lambda r}\|_{p/(\lambda r);\Omega} \|u^{(1-\lambda)r}\|_{q/(1-\lambda)r;\Omega}$$

for $0 \le \lambda \le 1$ and $\frac{1}{r} = \frac{\lambda}{p} + \frac{1-\lambda}{q}$. Noting that if $p \le r \le q$, one can always find such a λ, we have

$$\|u\|_{r;\Omega} \le \|u\|_{p;\Omega}^{\lambda} \|u\|_{q;\Omega}^{1-\lambda} \tag{8}$$

and proving the claim. □

The first property above tells us that, after *localising*, higher L^p norms control lower ones, and in particular, higher L^p norms have *more regularity*, or that they are *less singular*. The second property tells us that one can *harmonically* interpolate between higher and lower L^p spaces to get something in between.

Remark 2. The Hölder inequality can be iterated to obtain that, if $\sum_{i=1}^{m} (p_i)^{-1} = 1$, then

$$\int_{\Omega} f_1 f_2 \cdots f_m \, \mathrm{d}x \le \prod_{i=1}^{m} \|f_i\|_{p_i;\Omega}$$

The localisation above can be interpreted as a cut-off by the characteristic function χ_Ω. There are other ways to achieve localisation; a very useful one is to use *polynomial weights*.

Exercise 5. Let $L^p_s(\mathbb{R}^d)$ denote the Lebesgue space with s weight: that is, $f \in L^p_s(\mathbb{R}^d)$ iff $\int |f(x)|^p (1 + |x|^s)^p \, \mathrm{d}x < \infty$. Show that $L^q_s(\mathbb{R}^d)$ embeds continuously into $L^p(\mathbb{R}^d)$ if and only if $q > p$ and $s > \frac{1}{p} - \frac{1}{q}$. (Hint: $\Leftarrow$ follows from judicious application of Hölder inequality; $\Rightarrow$ follows by considering functions of the form $|x|^{-t}(1 + |x|^r)^{-1}$.)

Hölder's inequality can also be used to obtain the following useful inequality.

Lemma 3 (Minkowski's inequality). *Let $u(x, y)$ be a measurable function defined on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. Then provided both sides of the inequality are finite, we have, for $1 \le q \le p < \infty$,*

$$\left(\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} |u(x,y)|^q \, \mathrm{d}x \right)^{p/q} \mathrm{d}y \right)^{1/p} \le \left(\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} |u(x,y)|^p \, \mathrm{d}y \right)^{q/p} \mathrm{d}x \right)^{1/q} . \tag{9}$$

Remark 4. Note that the "$p = \infty$" case in the above inequality is trivially true.

Proof. Note that by replacing u by $|u|^q$, it suffices to prove the inequality for $q = 1$. The case $p = q = 1$ is then just Fubini's theorem. For $p > 1$, let

$$S(y) := \int_{\mathbb{R}^{d_1}} |u(x,y)| \, \mathrm{d}x$$

we re-write

$$S(y)^p = \int_{\mathbb{R}^{d_1}} |u(z,y)| \, \mathrm{d}z S(y)^{p-1} .$$

In this notation,

$$\begin{aligned}\int_{\mathbb{R}^{d_2}} S(y)^p \, \mathrm{d}y &= \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} |u(z,y)| \, \mathrm{d}z S(y)^{p-1} \, \mathrm{d}y \\ \text{(Fubini)} &= \int_{\mathbb{R}^{d_1}} \int_{\mathbb{R}^{d_2}} |u(z,y)| S(y)^{p-1} \, \mathrm{d}y \, \mathrm{d}z \\ \text{(Hölder)} &= \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} |u(z,y)|^p \, \mathrm{d}y \right)^{1/p} \mathrm{d}z \cdot \left(\int_{\mathbb{R}^{d_2}} S(w)^{(p-1)p'} \, \mathrm{d}w \right)^{1/p'} .\end{aligned}$$

Now, $(p-1)p' = p$ by definition. So we have that, divident both sides by a suitable factor of $\int S(w)^p \, \mathrm{d}w$, that

$$\left(\int_{\mathbb{R}^{d_2}} S(y)^p \, \mathrm{d}y \right)^{1-1/p'} \leq \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} |u(z,y)|^p \, \mathrm{d}y \right)^{1/p} \mathrm{d}z$$

precisely as claimed. □

1.2. **Friedrichs mollifiers, regularisation.** Now we discuss a systematic way to approximate L^p functions by smooth ones, due to K. O. Friedrichs.

Definition 5. A Friedrichs *mollifier*, or an *approximation to the identity*, is a non-negative function ψ in $C_c^\infty(B)$ (where $B = B_1(0)$ is the ball of radius 1 about the origin in $\mathbb{R}^d$) satisfying the condition that $\int_{\mathbb{R}^d} \psi \, \mathrm{d}x = 1$.

A standard example of a mollifier is the function

$$\psi(x) = \begin{cases} c \exp\left(\frac{1}{|x|^2 - 1}\right) & |x| \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

with c chosen so that the total integral of ψ is 1.

Now consider the expression for $\delta > 0$ defined by convolution

$$u_\delta(x) = \frac{1}{\delta^d} \int_{B_\delta(x)} \psi(\frac{x-y}{\delta}) u(y) \, \mathrm{d}y \, . \tag{10}$$

The right hand side of (10) is well-defined if u is defined, and integrable, on $B_\delta(x)$. We can treat it as a sort of weighted average over the small ball. Now, suppose $u \in L^p(\Omega)$. By Proposition 1, $u \in L^1_{\text{loc}}(\Omega)$. So for $\delta > 0$, as long as $dist(x, \partial\Omega) > \delta$, $u_\delta(x)$ is well-defined. Furthermore, using the property of the convolution, we see that for any $\Omega' \Subset \Omega$ with $dist(\Omega', \partial\Omega) > \delta$, we have $u_\delta \in C^\infty(\Omega')$. Also if $u \in L^p(\mathbb{R}^d)$ has compact support on $\mathbb{R}^d$, then $u_\delta \in C_c^\infty(\mathbb{R}^d)$.

Exercise 6. Verify that for $u \in L^p(\mathbb{R}^d)$, u_δ is also p-integrable. (Hint: by using Tonelli's theorem, it reduces to checking measurability of $u(x-y)$ on $\mathbb{R}^d \times \mathbb{R}^d$.)

The importance of the mollifier is not just that it smooths out a function, but that it also approximates it as $\delta \to 0$. Heuristically, as $\delta \to 0$ the expression $\delta^{-d}\psi(x/\delta)$ approaches the Dirac delta function, and so $u_\delta \to u$. To make it precise:

Lemma 6. *For $u \in L^p(\Omega)$, $p < \infty$, then $u_\delta \to u$ in $L^p(\Omega)$ as $\delta \to 0$.*

Proof. In the case where $\Omega \neq \mathbb{R}^d$, we extend u trivially by setting $u = 0$ outside Ω. Then $u \in L^p(\mathbb{R}^d)$. Since $\|u_\delta - u\|_{p;\Omega} \leq \|u_\delta - u\|_{p;\mathbb{R}^d}$, it suffices to show the convergence for $\Omega = \mathbb{R}^d$.

$$\begin{aligned} \|u_\delta - u\|_p &= \left(\int \left| \int (u(x-\delta y) - u(x))\psi(y)\, \mathrm{d}y \right|^p \mathrm{d}x \right)^{1/p} \\ \text{(Minkowski's inequality)} \quad &\leq \int \psi(y) \left(\int |u(x-\delta y) - u(x)|^p \, \mathrm{d}x \right)^{1/p} \mathrm{d}y \\ \text{(Hölder)} \quad &\leq \sup_{|y| \leq \delta} \|u(\cdot - y) - u\|_p \end{aligned}$$

That the right hand side of the last inequality $\searrow 0$ as $\delta \to 0$ can be checked by using the fact that the continuous functions with compact support are dense in L^p for any $1 \leq p < \infty$. $\square$

As a corollary, we have that the smooth functions are dense inside $L^p(\Omega)$.

Exercise 7. Find a counterexample to Lemma 6 in the case $p = \infty$.

1.3. **Weak derivatives.** First recall the formula for integrating by parts. Let $u \in C^{|\alpha|}(\Omega)$, and $\phi \in C_0^{|\alpha|}(\Omega)$, then we have

$$\int_\Omega (\partial^\alpha u)\phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(\partial^\alpha \phi) \, \mathrm{d}x \,.$$

Using this formula we can define a generalised notion of derivatives.

Definition 7. Let u be locally integrable in Ω. Then a locally integrable function v is said to be the α^{th} *weak derivative* of u, written $v = D^\alpha u$, if for every function $\phi \in C_0^{|\alpha|}(\Omega)$, we have

$$\int_\Omega v\phi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega u(\partial^\alpha \phi) \, \mathrm{d}x \,. \tag{11}$$

Note that when the weak derivative $D^\alpha u$ exists, it is defined only up to a set of measure zero. So any point-wise statements to be made about $D^\alpha u$ is understood to only hold almost surely.

Definition 8. A locally integrable function u is said to be *k-times weakly differentiable* if for all $|\alpha| \leq k$, the weak derivative $D^\alpha u$ exists.

Observe that a C^k function is automatically k-times weakly differentiable, the classical derivatives being representatives of the equivalence class of weak derivatives.

Exercise 8. (Derivatives and mollifiers commute.) Let $u \in L^1_{\text{loc}}(\Omega)$, and assume $D^\alpha u$ exists. Prove that for $x \in \Omega$ and $\delta > 0$ such that $dist(x, \partial\Omega) > \delta$, we have

$$\partial^\alpha u_\delta(x) = (D^\alpha u)_\delta(x) \,. \tag{12}$$

Exercise 9. (Approximation theorem for weak derivatives.) Let $u, v \in L^1_{\text{loc}}(\Omega)$, then $v = D^\alpha u$ if and only if $\exists$ a sequence (u_m) of $C^\infty(\Omega)$ functions such that $u_m \to u$ and $\partial^\alpha u_m \to v$ in $L^1_{\text{loc}}(\Omega)$.

By the approximate theorem, most of classical differential calculus can be reproduced for weak derivatives. For example, the product rule

$$D(uv) = (Du)v + u(Dv)$$

holds for every pair of weakly differentiable functions u, v as long as uv and $(Du)v + u(Dv)$ are both L^1_{loc}. Similarly, the chain rule holds: let $u \in L^1_{\mathrm{loc}}(\Omega)$ be weakly differentiable, let $y = y(x)$ be a C^1 coordinate change $\Omega' \to \Omega$, and let $f \in C^1(\mathbb{R})$ with f' bounded. Then (1) the function $v = u \circ y$ is weakly differentiable on Ω', with $Dv = Du \frac{\partial y}{\partial x}$; (2) the function $f \circ u$ is weakly differentiable on Ω, with $D(f \circ u) = f'(u) Du$. For more details on the differential calculus, please refer to Evans or Gilbarg-Trudinger.

1.4. **Sobolev spaces.**

We begin straight with the definition.

Definition 9. The notation $W^{k,p}(\Omega)$ is the *Sobolev space* of differentiability k and integrability p. It consists of functions u which are k-weakly differentiable, such that $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$.

The Sobolev spaces $W^{k,p}(\Omega)$ are Banach spaces with the norm

$$\|u\|_{p,k;\Omega} = \left(\int_\Omega \sum_{|\alpha| \leq k} |D^\alpha u|^p \, \mathrm{d}x \right)^{1/p} . \tag{13}$$

Observe that the space $W^{0,p}(\Omega)$ is just $L^p(\Omega)$. In the case that $p = 2$, we also introduce the notation $H^k(\Omega) = W^{k,2}(\Omega)$. These L^2-Sobolev spaces are Hilbert spaces under the inner product

$$\langle u, v \rangle_k = \int_\Omega \sum_{|\alpha| \leq k} D^\alpha u D^\alpha v \, \mathrm{d}x . \tag{14}$$

> These L^2 Sobolev spaces will be used heavily in Mihalis' class; one particular reason is that Hilbert space techniques are very powerful in demonstrating existence of solutions to partial differential equations, another is that for hyperbolic PDEs, L^2-based spaces are the only ones in which we can do inductive arguments.

The localised versions of these spaces are defined analogously to the case of Lebesgue spaces. There are two more variants of the Sobolev spaces that are commonly used.

Definition 10. The spaces $\mathring{W}^{k,p}(\Omega)$ (similarly $\mathring{H}^k(\Omega)$) are the *homogeneous Sobolev spaces*. They consist of k-times weakly differentiable functions u such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| = k$.

Definition 11. The spaces $W^{k,p}_0(\Omega)$ (similarly $H^k_0(\Omega)$) are the closure of $C^k_0(\Omega)$ under the Sobolev norm (13).

The $\mathring{W}^{k,p}$ spaces are not Banach spaces; the defining condition does not give a norm, it merely gives a semi-norm. For example, two functions in C^k that differ by a constant are the "same size" under this semi-norm. We can "mod out" this ambiguity by requiring that the representatives we consider has mean 0 over Ω, as long as $|\Omega| < \infty$.

A common reason to consider the homogeneous Sobolev spaces is that, if the domain $\Omega = \mathbb{R}^d$, it has nice scaling properties. See §2.6.

Remark 12. The $W^{k,p}$ and $W_0^{k,p}$ spaces are generally not equal, except when $\Omega = \mathbb{R}^d$.

Using the approximation theorem for weak derivatives and the commutation relation, we see immediately that for $u \in W^{k,p}(\Omega)$, the mollified $\partial^\alpha u_\delta \to D^\alpha u$ as $\delta \to 0$ in $L^p_{\text{loc}}(\Omega)$. Here we derive a global approximation.

Theorem 13. *The subspace $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$, for $p < \infty$.*

Proof. Let Ω_i be an approximation of Ω by compactly included subsets; that is, $\Omega_i \Subset \Omega_{i+1}$, $\cup_i \Omega_i = \Omega$, and $\Omega_j = \emptyset$ for $j \leq 0$. Let η_j be a partition of unity subordinate to the covering $\{\Omega_{j+1} \setminus \Omega_{j-1}\}$. For $u \in W^{k,p}(\Omega)$, and for any $\epsilon > 0$, using the result mentioned before the statement of the theorem, we can choose a sequence of δ_j such that the following are satisfied:

$$\delta_j < dist(\Omega_{j+1}, \partial\Omega_{j+3})$$
$$\left\| (\eta_j u)_{\delta_j} - \eta_j u \right\|_{p,k;\Omega} \leq \frac{\epsilon}{2^{j+1}}$$

Let $v = \sum (\eta_j u)_{\delta_j}$. By the definition of partition of unity, and the first condition above, we have that at each $x \in \Omega$, only finitely many terms in the infinite sum is non-zero. So $v \in C^\infty(\Omega)$ by construction. Furthermore, using the triangle inequality

$$\|u - v\|_{p,k;\Omega} \leq \sum \left\| (\eta_j u)_{\delta_j} - \eta_j u \right\|_{p,k;\Omega} \leq \epsilon$$

and we obtain the approximation. $\square$

In view of the Density Theorem 13, Definitions 9 and 11 can be re-written to have the Sobolev spaces defined as the completion of $C^\infty(\Omega)$ and $C_c^\infty(\Omega)$ respectively in the $W^{k,p}(\Omega)$ norm. Note that in the case $\Omega = \mathbb{R}^d$, the spaces $W^{k,p}$ and $W_0^{k,p}$ coïncide, and thus $C_c^\infty(\mathbb{R}^d)$ is dense also in $W^{k,p}(\Omega)$.

1.5. **Difference quotients.** The classical partial derivative is defined, for a smooth function u, as

$$\partial_i u(x) = \lim_{h \to 0} \frac{u(x + he_i) - u(x)}{h} .$$

For considering weak derivatives, we use a similar idea called *difference quotients*.

Definition 14. Given a unit vector v and some measurable function u, the *h-translate of u in the direction of v* is denoted as $\tau_v^h u$, and is defined by

$$\tau_v^h u(x) = u(x + vh) .$$

Definition 15. For a measurable function u, the *difference quotient of u in the coordinate direction e_i of length $h \neq 0$* is the measurable function denoted by $\Delta_i^h u$ and defined by

$$\Delta_i^h u = \frac{\tau_{e_i}^h u - u}{h} .$$

Exercise 10. Derive the product rule for Δ_i^h.

The key point of this construction, unlike that of the weak derivative, is that for an arbitrary locally integrable u, $\Delta_i^h u$ always exists and is locally integrable, whereas $D_i u$ may not be defined. The two notions, however, are intimately connected.

Proposition 16. *Let $u \in W^{1,p}(\Omega)$. Then $\Delta^h u \in L^p(\Omega')$ for any $\Omega' \Subset \Omega$ with $|h| < dist(\Omega', \partial\Omega)$. That is,*

$$\|\Delta^h u\|_{p;\Omega'} \leq \|Du\|_{p;\Omega} .$$

Proof. By a density argument, if suffices to prove the inequality for $u \in C^1(\Omega)$, since it is clear that $\Delta^h u_\delta = (\Delta^h u)_\delta$. Notice that the restriction to Ω' is so that the difference quotient is well-defined.

For continuously differentiable functions, the difference quotient can be written as

$$\Delta_i^h u(x) = \frac{1}{h} \int_0^h \partial_i u(x + te_i) \, \mathrm{d}t .$$

Now using the first part of Proposition 1, we have

$$|\Delta_i^h u(x)| \leq \frac{1}{h} \int_0^h |\partial_i u(x + te_i) \, \mathrm{d}t \leq \frac{1}{h^{1/p}} \left(\int_0^h |\partial_i u(x + te_i)|^p \, \mathrm{d}t \right)^{1/p} .$$

So integrating in $L^p(\Omega')$ we have

$$\|\Delta_i^h u\|_{p;\Omega'}^p \leq \frac{1}{h} \int_0^h \|\tau_{e_i}^t \partial_i u\|_{p;\Omega'}^p \, \mathrm{d}t \leq \frac{1}{h} \int_0^h \, \mathrm{d}t \|\partial_i u\|_{p;\Omega}^p$$

and the desired inequality follows. □

The implication is also allowed to run the other way.

Proposition 17. *Let $u \in L^p(\Omega)$ for $1 < p < \infty$. If there exists some constant K such that for all $\Omega' \Subset \Omega$ and $0 < h < dist(\Omega', \partial\Omega)$, we have $\|\Delta^h u\|_{p;\Omega'} \leq K$, then u is weakly differentiable and $\|Du\|_{p;\Omega} \leq K$.*

Proof. Let $h_m \searrow 0$ be an arbitrary decreasing sequence, and $\Omega_m \nearrow \Omega$ such that $dist(\Omega_m, \partial\Omega) > h$. Then since $\|\Delta^{h_m} u\|_{p;\Omega_m} \leq K$, we have that for each direction e_i there exists $v_i \in L^p(\Omega)$ such that the weak convergence

$$\Delta_i^{h_m} u \rightharpoonup v_i$$

in $L^p(\Omega')$ holds for all $\Omega' \Subset \Omega$, with $\|v_i\|_{p;\Omega} \leq K$. In particular, this implies that for any $\phi \in C_c^1(\Omega)$,

$$\int_\Omega \phi \Delta_i^{h_m} u \, \mathrm{d}x \to \int_\Omega \phi v_i \, \mathrm{d}x .$$

For m sufficiently large, $h_m < dist(\operatorname{supp}\phi, \partial\Omega)$, so we can "integrate by parts":

$$\begin{aligned}\int_\Omega \phi \Delta_i^{h_m} u \,\mathrm{d}x &= \frac{1}{h_m}\left(\int_\Omega \phi \tau_{e_i}^{h_m} u \,\mathrm{d}x - \int_\Omega \phi u \,\mathrm{d}x\right) \\ &= \frac{1}{h_m}\left(\int_\Omega u \tau_{e_i}^{-h_m} \phi \,\mathrm{d}x - \int_\Omega \phi u \,\mathrm{d}x\right) \\ &= -\int_\Omega u \Delta_i^{-h_m} \phi \,\mathrm{d}x \,.\end{aligned}$$

Since ϕ is classically differentiable, taking the limit $h_m \to 0$ gives convergence

$$\int_\Omega \phi \Delta_i^{h_m} u \,\mathrm{d}x \to -\int_\Omega u \partial_i \phi \,\mathrm{d}x \,.$$

So

$$\int_\Omega \phi v_i \,\mathrm{d}x = -\int_\Omega u \partial_i \phi$$

for all $\phi \in C_c^1(\Omega)$, so $v_i = D_i u$. □

2. Imbedding Theorems and Friends

In this second part, when we write Ω, it always denotes a *bounded, connected, and open* subset of $\mathbb{R}^d$. Some results in the sequel pertains to the domain being $\mathbb{R}^d$ itself; they will be clearly marked as such.

In the following we will only work with Sobolev spaces that vanish on the boundary of Ω (for the domain being $\mathbb{R}^d$, this condition is trivial). Quite a few of the theorems presented below will not be true in general if we consider $W^{k,p}(\Omega)$ associated to arbitrary Ω. This failure is usually due to our inability to arbitrarily extend a $W^{k,p}$ function across $\partial\Omega$ in general. In the cases where $\partial\Omega$ has nice regularity properties (external cone condition, differentiability), these theorems can usually be salvaged. The reader is advised to consult the comprehensive volume of Adams for these cases.

2.1. **Gagliardo-Nirenberg-Sobolev inequality.**

The Gagliardo-Nirenberg-Sobolev inequality is an inequality for the domain being $\mathbb{R}^d$.

Theorem 18 (Gagliardo-Nirenberg-Sobolev)**.** *For $u \in C_c^\infty(\mathbb{R}^d)$, for any $1 \le p < d$, there exists a constant C depending only on p and d such that*

$$\|u\|_{\frac{dp}{d-p};\mathbb{R}^d} \le C\|\partial u\|_{p;\mathbb{R}^d} \tag{15}$$

Note that by density, the above inequality extends to $u \in W^{k,p}(\mathbb{R}^d)$. And by the interpolation inequality in Proposition 1, this implies that (since $dp/(d-p) > p$) for any $p \le q \le \frac{dp}{d-p}$, there is a continuous embedding $W^{1,p}(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$.

Proof. First we reduce it to the case where $p = 1$. Let $v = (u^2)^\gamma$ for $\gamma > \frac{1}{2}$. Then $|\partial v| = 2\gamma(u^2)^{\gamma-1}|u\partial u|$, and notice that for $u \in C^1$, $\partial v = 0$ when $u \to 0$: so $v \in C_c^1(\mathbb{R}^d)$.

Using the density of $C_c^\infty \subset C_c^1$, we assume the theorem holds for v with $p = 1$, that is,

$$\|v\|_{\frac{d}{d-1};\mathbb{R}^d} \leq C_{1,d}\|\partial v\|_{1;\mathbb{R}^d} .$$

Plugging in the definition we have

$$\|u^{2\gamma}\|_{\frac{d}{d-1};\mathbb{R}^d} \leq 2\gamma C_{1,d}\|u^{2\gamma-2}u\partial u\|_{1;\mathbb{R}^d} \leq 2\gamma C_{1,d}\|u^{2\gamma-2}u\|_{p';\mathbb{R}^d}\|\partial u\|_{p;\mathbb{R}^d}$$

using Hölder inequality again. Then we solve

$$\frac{2\gamma d}{d-1} = (2\gamma - 1)p' \implies 2\gamma = p\frac{d-1}{d-p} .$$

When we plug it in we have

$$\|u\|^{2\gamma}_{\frac{dp}{d-p};\mathbb{R}^d} \leq p\frac{d-1}{d-p}C_{1,d}\|u\|^{2\gamma-1}_{\frac{dp}{d-p};\mathbb{R}^d}\|\partial u\|_{p;\mathbb{R}^d} .$$

Cancelling out the redundant factors leads us to (15) as desired.

It remains to prove the inequality for the case $p = 1$. Observe that for each $1 \leq i \leq d$,

$$|u(x)| \leq \int_{-\infty}^{x_i} |\partial_i u(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_d)| \,\mathrm{d}y$$

so we can write

$$|u(x)|^{\frac{d}{d-1}} \leq \left(\prod_{i=1}^{d}\int_{-\infty}^{x_i} |\partial_i u| \,\mathrm{d}y_i\right)^{\frac{1}{d-1}} .$$

We want to integrate both sides over $\mathbb{R}^d$, and use the iterated Hölder inequality (see Remark 2). To do so we first notice that

$$\int_{\mathbb{R}} \left(\prod_{i=1}^{d}\int_{-\infty}^{x_i} |\partial_i u| \,\mathrm{d}y_i\right)^{\frac{1}{d-1}} \mathrm{d}x_j = \int_{\mathbb{R}} \underbrace{\left(\prod_{i\neq j}\int_{-\infty}^{x_i} |\partial_i u| \,\mathrm{d}y_i\right)^{\frac{1}{d-1}}}_{(d-1)\,\times\, L^{d-1}} \underbrace{\left(\int_{-\infty}^{x_j} |\partial_j u| \,\mathrm{d}y_j\right)^{\frac{1}{d-1}}}_{1\,\times\, L^{\infty}} \mathrm{d}x_j$$

and

$$\sup_{x_j}\left(\int_{-\infty}^{x_j} |\partial_j u| \,\mathrm{d}y_j\right)^{\frac{1}{d-1}} = \left(\int_{\mathbb{R}} |\partial_j u| \,\mathrm{d}y_j\right)^{\frac{1}{d-1}}$$

and that by taking the L^{d-1} norm of the other terms, we can pass the integral inside

$$\left\|\left(\int_{-\infty}^{x_i} |\partial_i u| \,\mathrm{d}y_i\right)^{\frac{1}{d-1}}\right\|_{d-1;-\infty<x_j<\infty, j\neq i} = \left(\int_{-\infty}^{x_i}\int_{\mathbb{R}} |\partial_i u| \,\mathrm{d}x_j \,\mathrm{d}y_i\right)^{\frac{1}{d-1}} .$$

So we finally arrive at the inequality

$$\int_{\mathbb{R}^d} |u|^{\frac{d}{d-1}} \,\mathrm{d}x \leq \left(\prod_{i=1}^{d}\int_{\mathbb{R}^d} |\partial_i u| \,\mathrm{d}x\right)^{\frac{1}{d-1}}$$

or

$$\|u\|_{\frac{d}{d-1};\mathbb{R}^d} \leq \left(\prod_{i=1}^{d} \int_{\mathbb{R}^d} |\partial_i u| \, \mathrm{d}x \right)^{\frac{1}{d}} .$$

Now, applying the arithmetic-geometric-mean inequality, we get

$$\|u\|_{\frac{d}{d-1};\mathbb{R}^d} \leq \frac{1}{d} \sum_i \|\partial_i u\|_{1;\mathbb{R}^d} = \frac{1}{d} \|\partial u\|_{1;\mathbb{R}^d}$$

as claimed. And thus we've proven the theorem with $C = \frac{p(d-1)}{d(d-p)}$. □

Remark 19. For dimension $d = 1$, if the derivative of a function is absolutely integrable, the function is absolutely continuous and bounded. In other words, we have

$$\|u\|_{\infty;\mathbb{R}} \leq \|\partial u\|_{1;\mathbb{R}} .$$

For higher dimensions, the end-point case of $p = d$ is not true. The easiest counterexample is constructed via the Fourier transform (see Remark 20 below).

The Gagliardo-Nirenberg-Sobolev theorem can be iterated. The easiest way to remember the exponents is perhaps the following observation:

$$\left(\frac{dp}{d-p} \right)^{-1} = p^{-1} - d^{-1} . \tag{16}$$

And so if you are willing to lose k derivatives, you have

$$\|u\|_{\frac{dp}{d-kp};\mathbb{R}^d} \leq C_{p,d,k} \sum_{|\alpha|=k} \|\partial^\alpha\|_{p;\mathbb{R}^d}$$

for any $u \in C_c^\infty(\mathbb{R}^d)$.

Remark 20 (Counterexample to L^∞ endpoint Sobolev). Let us focus now in the case of L^2-Sobolev spaces H^k. We will see that in the case $2k = d$, we do not have the embedding $H^k \in L^\infty$. Using the fact that the Fourier transform acts on L^2 by isometry (i.e. Plancherel's theorem that $\int |f|^2 \, \mathrm{d}x = \int |\hat{f}|^2 \, \mathrm{d}\xi$), we see that the space H^k is equivalently characterized as all L^2 functions u such that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\hat{u}(\xi)|^2 \, \mathrm{d}\xi < \infty .$$

Now let

$$\hat{u}(\xi) = \frac{1}{(1 + |\xi|^2)^k \log(2 + |\xi|^2)} .$$

$\hat{u}$ is clearly in L^2 for $k > \frac{d}{4}$, and thus is the Fourier transform of some L^2 function. And similarly $(1 + |\xi|^2)^{k/2} \hat{u}$ is in L^2 if $k > \frac{d}{2}$. For $2k = d$,

$$\int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^k [\log(2 + |\xi|^2)]^2} \, \mathrm{d}\xi \leq C_1 + C_2 \int_1^\infty r^{-1} (\log r)^{-2} \, \mathrm{d}r ,$$

by converting to polar coordinates, and using that the function is smooth in the ball of radius 1. Noting that $\frac{d}{dr}(\log r)^{-1} = -r^{-1}(\log r)^{-2}$, we have that u represents a function in $H^k(\mathbb{R}^d)$ for $2k = d$.

On the other hand, using the definition of the inverse Fourier transform, $u(0) = \int \hat{u}\, \mathrm{d}\xi$. A polar coordinate change shows that the right hand side is bounded below by some constant times $\int_1^\infty (r \log r)^{-1}\, \mathrm{d}r$, but with the antiderivative of $(r \log r)^{-1}$ being $\log\log r$, the integral diverges. This is why we say that the Gagliardo-Nirenberg-Sobolev inequality diverges logarithmically as $kp \to d$.

As remarked earlier, Gagliardo-Nirenberg-Sobolev inequality only holds for the precise Lebesgue exponent stated; one can get a range of exponents from interpolation on unbounded domains. On bounded domains, we can use the first part of Proposition 1 and remove the lower bound of $p \leq q$. The following in particular implies Poincaré's inequality.

Corollary 21 (Sobolev inequality). *For bounded open domain $\Omega \subset \mathbb{R}^d$, we have $W_0^{k,p}(\Omega) \to L^q(\Omega)$ if $1 \leq q \leq \frac{dp}{d-kp} < \infty$. More precisely, for $u \in W_0^{k,p}(\Omega)$ and q as above,*

$$\|u\|_{q;\Omega} \leq C \sum_{|\alpha|=k} \|D^\alpha u\|_{p;\Omega} .$$

Proof. This follows from Theorem 18, the density of $C_c^\infty(\Omega)$ in $W_0^{k,p}(\Omega)$, and Proposition 1. □

In general, any inequality which trades differentiability for integrability is called a Sobolev or Sobolev-type inequality in the literature. Note that the trade is one-way. One cannot generally sacrifice integrability to gain differentiability, as locally integrability functions need not have a well-defined weak derivative.

2.2. **Morrey's inequality.** Thus far we have considered the embedding theorems for $kp < n$. What happens when $kp > n$? We know that when $kp = n$ there is a logarithmic divergence, but often this type of divergences form a boundary between two different regimes. (For example, the function $1/x$ diverges logarithmically under integration, but is the boundary between "locally integrable" and "integrable except for local defects".) Indeed, for $kp > n$ there is a different class of estimates, historically attributed to Morrey. Here we will prove a slightly weakened version, originally due to Sobolev.

Theorem 22 (Morrey-Sobolev). *For $u \in C_c^\infty(\mathbb{R}^d)$, then there exists a constant C depending on p and d such that*

$$\sup|u| \leq C |\operatorname{supp} u|^{\frac{1}{d}-\frac{1}{p}} \|\partial u\|_{p;\mathbb{R}^d} \tag{17}$$

if $p > d \geq 2$.

That the support of u comes into play is natural. By rescaling the spatial variables, we can make the support larger and larger while making the function "flatter and flatter", and at the same time keeping the maximum of u unchanged. So without the $\operatorname{supp} u$ term, we can make the right hand side as small as we want, while keeping the left hand side fixed, contradicting the inequality.

Proof. Perform the rescaling where $u'(x) = u(\lambda x)$, then $\partial u'(x) = \lambda(\partial u)(\lambda x)$. And $\|\partial u'\|_p = \lambda^{1-\frac{d}{p}}\|\partial u\|_p$. On the other hand $|\operatorname{supp} u'| = \lambda^{-d}|\operatorname{supp} u|$, so the inequality to prove is invariant under rescaling of the spatial variables. Therefore it suffices to prove the inequality for functions u such that $|\operatorname{supp} u| = 1$. Furthermore, by rescaling $u \to \lambda u$, we see that the inequality is also invariant, and so we can assume $\|\partial u\|_p = 1$. It then suffices to show that there exists some constant C depending on p and d such that $\sup|u| \leq C$ under these assumptions.

Now observe that $p > d \geq 2$, so $p' = \frac{p}{p-1} < d' = \frac{d}{d-1} \leq 2$. Fix $\gamma > 1$, we first make use of the Gagliardo-Nirenberg-Sobolev inequality

$$\||u|^\gamma\|_{d';\mathbb{R}^d} \leq \gamma\||u|^{\gamma-1}\partial u\|_{1;\mathbb{R}^d} .$$

Applying Hölder inequality to the right hand side, we get

$$\|u\|^\gamma_{\gamma d';\mathbb{R}^d} \leq \gamma\|u\|^{\gamma-1}_{(\gamma-1)p';\mathbb{R}^d}\|\partial u\|_{p;\mathbb{R}^d} ,$$

where, by our assumption, the last factor in the right hand side is 1. Now, since $(\gamma-1)p' \leq \gamma p'$, and that u is supported on a set of measure 1, we can use the first part of Proposition 1 and arrive at

$$\|u\|_{\gamma d';\mathbb{R}^d} \leq \gamma^{1/\gamma}\|u\|^{1-\frac{1}{\gamma}}_{\gamma p';\mathbb{R}^d}$$

Now take $\eta = d'/p' > 1$ and set $\gamma = \eta^k$. For $k > 0$, we have

$$\|u\|_{\eta^k d';\mathbb{R}^d} \leq \eta^{k/\eta^k}\|u\|^{1-\frac{1}{\eta^k}}_{\eta^{k-1}d';\mathbb{R}^d} .$$

For the base case $k = 0$ we use the Gagliardo-Nirenberg-Sobolev inequality again, together with Hölder's inequality to get

$$\|u\|_{d';\mathbb{R}^d} \leq \|\partial u\|_{1;\mathbb{R}^d} \leq |\operatorname{supp} u|^{1-1/p}\|\partial u\|_{p;\mathbb{R}^d} = 1 .$$

And so iterating, we have

$$\|u\|_{\eta^k d';\mathbb{R}^d} = \|u\|_{\eta^k d';\operatorname{supp} u} \leq \eta^{\sum j\eta^{-j}} = C(p,d) .$$

Lastly we apply the fact that

$$\lim_{p\to\infty}\left(\frac{1}{|\Omega|}\int_\Omega |u|^p \,\mathrm{d}x\right)^{1/p} = \sup_\Omega|u| \tag{18}$$

for any bounded domain Ω, we have that

$$\sup|u| = \sup_{\operatorname{supp} u}|u| \leq C$$

as claimed. □

Exercise 11. Prove (18).

In Theorem 22, we see that the fact u has compact support is essentially used. This means that only knowing the derivative of a function is p-integrable does not guarantee the function itself is bounded. A simple example is obtained by considering $u(x) = (1+|x|)^{1/3}$ in one dimension. It has one derivative that is L^p integrable for all $p \geq 2$, but it is not a bounded function.

However, all is not lost! If we also know that the function itself is in L^p, we can "break the scaling" and get control on boundedness. In the proof we will also see an illustration of the useful technique in harmonic/functional analysis called *optimisation*.

Corollary 23. *For $u \in C^\infty(\mathbb{R}^d)$ and $p > d$, there exists a constant C depending on p and d such that*

$$\sup |u| \leq C \|u\|_{p;\mathbb{R}^d}^{1-\frac{d}{p}} \|\partial u\|_{p;\mathbb{R}^d}^{\frac{d}{p}} . \tag{19}$$

Proof. Let $t > 0$ be arbitrary, and denote by Ω_t the set $\{|u| > t\}$. Let v be the function such that $v = 0$ outside Ω_t and $v = |u| - t$ on Ω_t. Clearly $v \in W_0^{1,p}(\Omega_t)$. Using that u is smooth, $v = \pm u \mp t$ on each of the connected components of Ω_t. Therefore $\|\partial v\|_{p;\Omega_t} \leq \|\partial u\|_{p;\mathbb{R}^d}$. Observe that

$$\sup |u| \leq t + \sup |v| .$$

Applying Theorem 22 we get

$$\sup |v| \leq C |\Omega_t|^{\frac{1}{d} - \frac{1}{p}} \|\partial v\|_{p;\Omega_t} .$$

We can estimate the size of Ω_t using the L^p-weak-L^p estimate (also known as Chebyshev's inequality or Markov's inequality) which states that

$$|\Omega_t| \leq \frac{1}{t^p} \|u\|_{p;\mathbb{R}^d}^p .$$

Thus we arrive at

$$\sup |v| \leq C t^{1-\frac{p}{d}} \|u\|_{p;\mathbb{R}^d}^{\frac{p}{d}-1} \|\partial u\|_{p;\mathbb{R}^d}$$

which implies

$$\sup |u| \leq t \left(1 + C t^{-\frac{p}{d}} \|u\|_{p;\mathbb{R}^d}^{\frac{p}{d}-1} \|\partial u\|_{p;\mathbb{R}^d} \right) . \tag{20}$$

We now *optimise* by choosing t as a function of $\|u\|_{p;\mathbb{R}^d}$ and $\|\partial u\|_{p;\mathbb{R}^d}$. In particular we can choose t such that the term inside the parenthesis in (20) is a constant: we set

$$C \|u\|_{p;\mathbb{R}^d}^{\frac{p}{d}-1} \|\partial u\|_{p;\mathbb{R}^d} = t^{\frac{p}{d}} .$$

This implies that

$$\sup |u| \leq 2 C^{\frac{d}{p}} \|u\|_{p;\mathbb{R}^d}^{1-\frac{d}{p}} \|\partial u\|_{p;\mathbb{R}^d}^{\frac{d}{p}}$$

as claimed. □

Corollary 24. *(1) For $p > d$, the inclusion $W_0^{1,p}(\Omega) \to C(\bar{\Omega})$ is continuous. (2) Similarly the inclusion $W_0^{k,p}(\Omega) \to C^l(\bar{\Omega})$ for $0 \leq l < k - d/p$. (3) If $u \in W^{k,p}(\mathbb{R}^d)$, then for any Ω with compact closure we have $u \in C^l(\Omega)$ if $0 \leq l < k - d/p$.*

Exercise 12. Prove Corollary 24. (Hint: for $x, y \in \Omega' \Subset \Omega$, consider the function $v(y) = \psi(y)(u(y) - u(x))$ where $\psi(y) \in C_c^\infty(\Omega)$ and $\psi(y) = 1$ on Ω'. By letting $|\Omega'|$ and $|\operatorname{supp} v| \searrow 0$ one can recover continuity from Theorem 22.)

2.3. **Product estimates.** By putting together the Hölder inequality and the Sobolev inequalities, we get

Theorem 25 (Sobolev product estimates). *Let* $u_1 \in W^{k_1,p_1}(\mathbb{R}^d)$ *(or* $W_0^{k_1,p_1}(\Omega)$*) and* $u_1 \in W^{k_2,p_2}(\mathbb{R}^d)$ *(*$W_0^{k_2,p_2}(\Omega)$*). Then* $u_1 \cdot u_2 \in W^{k,p}(\mathbb{R}^d)$ *(*$W_0^{k,p}(\Omega)$*), whenever* $k \le \min(k_1, k_2)$ *and*

$$\frac{1}{p} - \frac{k}{d} > \frac{1}{p_1} - \frac{k_1}{d} + \frac{1}{p_2} - \frac{k_2}{d} \,. \tag{21}$$

In other words, there exists a constant $C = C(k_1, k_2, k, p_1, p_2, p, d)$ *such that for any* $u_1, u_2 \in C_c^\infty(\mathbb{R}^d)$,

$$\|u_1 u_2\|_{p,k;\mathbb{R}^d} \le C\|u_1\|_{p_1,k_1;\mathbb{R}^d}\|u_2\|_{p_2,k_2;\mathbb{R}^d} \,.$$

Proof. (Hereon C stands for a constant that may change line by line, but only depends on the parameters specified above.) By the product rule for differentiation,

$$\partial^\alpha(u_1 u_2) = \sum_{\beta+\gamma=\alpha} \partial^\beta u_1 \partial^\gamma u_2 \,.$$

So using the triangle inequality

$$\|u_1 u_2\|_{p,k;\mathbb{R}^d} \le \sum_{|\beta+\gamma|\le k} \|\partial^\beta u_1 \partial^\gamma u_2\|_{p;\mathbb{R}^d} \,.$$

For a pair (β, γ), we note that

$$\|\partial^\beta u_1 \partial^\gamma u_2\|_{p;\mathbb{R}^d} \le \|\partial^\beta u_1\|_{r;\mathbb{R}^d}\|\partial^\gamma u_2\|_{q;\mathbb{R}^d}$$

where $p^{-1} = q^{-1} + r^{-1}$ by Hölder. Using Sobolev inequality,

$$\|\partial^\beta u_1\|_{r;\mathbb{R}^d} \le C\|u\|_{p_1,k_1;\mathbb{R}^d}$$

when $k_1 > |\beta|$ and $q^{-1} > p_1^{-1} - \frac{k_1-|\beta|}{d}$. So

$$\|\partial^\beta u_1 \partial^\gamma u_2\|_{p;\mathbb{R}^d} \le C\|u_1\|_{p_1,k_1;\mathbb{R}^d}\|u_2\|_{p_2,k_2;\mathbb{R}^d}$$

whenever

$$\frac{1}{p} > \frac{1}{p_1} - \frac{k_1 - |\beta|}{d} + \frac{1}{p_2} - \frac{k_2 - |\gamma|}{d} \ge \frac{1}{p_1} - \frac{k_1}{d} + \frac{1}{p_2} - \frac{k_2}{d} + \frac{k}{d}$$

as claimed. □

A direct consequence is

Corollary 26. *The spaces* $W^{k,p}$ *for* $kp > d$ *are algebras, i.e. if* u_1, u_2 *are in* $W^{k,p}$*, so is the product* $u_1 \cdot u_2$*.*

2.4. **Rellich-Kondrachov compactness.** The continuous inclusion of one Banach space B_1 into another B_2 is said to be *compact* if the image of the unit ball in B_1 is precompact in B_2. That is, for every sequence $\{f_k\} \subset B_1$ with $\|f_k\|_{B_1} \le 1$, there exists a subsequence which is Cauchy in B_2. Compactness is particularly useful for *variational problems*: to minimize a functional, one is led to study a minimizing sequence. A suitable compactness result allows one to state that the minimizing sequence actually converges, though often in a less regular space (which is not as bad as it sounds, since quite often, especially in elliptic situations, we can gain back the regularity once we have a weak solution).

For a function space defined over subsets of $\mathbb{R}^d$, there are generally three ways for compactness to fail, which are all related to the symmetries of $\mathbb{R}^d$:

(1) divergence in *scale*;
(2) divergence in *physical support*; and
(3) divergence in *frequency support*.

The divergence in physical support is easiest to illustrate. Consider u a function with support contained within the ball of radius 1. Then for any translation invariant norm on $\mathbb{R}^d$ (any non-weighted Sobolev or Lebesgue norm will be one such), the sequence of functions $u_m(x) = u(x + 4me_1)$ is bounded, but has no Cauchy subsequence. This is because the supports of u_m, u_n for $m \neq n$ are disjoint, so $\|u_m - u_n\| = 2\|u\| \neq 0$.

On compact/bounded sets, a sequence cannot run off to infinity. But with a proper rescaling we can still have infinitely many functions with disjoint support fitting in the set. A classic example is given when $\Omega = (0,1) \subset \mathbb{R}$. In $L^1(\Omega)$, consider the sequence of functions

$$u_m(x) = 2^m \chi_{(2^{-m}, 2^{-m+1})}(x)$$

with $\|u_m\|_{1;(0,1)} = 1$. This sequence has again disjoint supports and thus contains no Cauchy subsequence. Another example is given by the failure of the Sobolev embedding $\mathring{W}_0^{1,p}(\Omega) \to L^{\frac{dp}{d-p}}(\Omega)$ to be compact.

Lemma 27. *The Sobolev embedding* $\mathring{W}_0^{1,p}(\Omega) \to L^{\frac{dp}{d-p}}(\Omega)$ *is not compact.*

Proof. Since Ω is open, it contains some metric ball. Without loss of generality, we assume it contains the unit ball B about the origin. Take a function u in $C_c^\infty(B)$ such that $u = 1$ on the ball of radius $1/2$. The functions in the sequence

$$u_m(x) = 2^{m(\frac{d}{p}-1)} u(2^m x)$$

can be shown by explicit computation to all have the same norms

$$\|u_m\|_{\frac{dp}{d-p};B} = \|u\|_{\frac{dp}{d-p};B}, \qquad \|\partial u_m\|_{p;B} = \|\partial u\|_{p;B}.$$

So this is a bounded sequence in $\mathring{W}_0^{1,p}(\Omega)$.

On the other hand, we can compute

$$\begin{aligned} \|u_m - u_n\|_{\frac{dp}{d-p};B} &= \|u_{|m-n|} - u\|_{\frac{dp}{d-p};B} \\ &\geq \left(2^{|m-n|(\frac{d}{p}-1)} - 1\right)\left(2^{-d-|m-n|d}|B|\right)^{\frac{d-p}{dp}} \\ &\geq 2^{1-\frac{d}{p}}\left(1 - 2^{1-\frac{d}{p}}\right)|B|^{\frac{d-p}{dp}} \end{aligned}$$

and so the sequence admits no Cauchy subsequence. □

Remark 28. Had we examined instead the imbedding $W_0^{1,p}(\Omega) \to L^q(\Omega)$ where $q < \frac{dp}{d-p}$, we would've found that the above scaling construction leads to $u_m \to 0$ in L^q.

The divergence in frequency support is similar to the divergence in physical support, except for a conjugation via the Fourier transform. Here I'll merely given an example. By Proposition 1, we have a continuous inclusion $L^q(\Omega) \to L^p(\Omega)$ is $p < q$.

Lemma 29. *The inclusion $L^q(\Omega) \to L^p(\Omega)$ is not compact.* [1]

Proof. Using that Ω is open, without loss of generality, we assume that the cube $[0,1]^d \Subset \Omega$. Consider the functions

$$u_m(x) = \begin{cases} \sin(2\pi m x_1) & x \in [0,1]^d \\ 0 & x \notin [0,1]^d \end{cases}$$

where are bounded on Ω and hence bounded in any L^q.

On the other hand, the interpolation inequality gives us

$$\|u_m - u_n\|_{2;\Omega}^2 \le \|u_m - u_n\|_{1;\Omega} \|u_m - u_n\|_{\infty;\Omega} \le 2\|u_m - u_n\|_{1;\Omega} \ .$$

While

$$\|u_m - u_n\|_{2;\Omega}^2 = \int\limits_{[0,1]^d} u_m^2 + u_n^2 - 2u_m u_n \, \mathrm{d}x = \int\limits_0^1 \sin(2\pi m y)^2 + \sin(2\pi n y)^2 \, \mathrm{d}y = 1 \ .$$

So the sequence has no Cauchy subsequence in $L^1(\Omega)$, and hence no Cauchy subsequence in $L^p(\Omega)$ for any p. □

So much for negative results. Here we'll give one positive result due to Rellich in the $p = 2$ case and Kondrachov in general.

Theorem 30 (Rellich-Kondrachov lemma)**.** *On a bounded open set Ω, the non-endpoint Sobolev imbeddings*

$$W_0^{1,p}(\Omega) \to L^q(\Omega)$$

where $q < \frac{dp}{d-p}$ is compact.

Proof. It suffices to prove it for $q = 1$. Then since the inclusion $W_0^{1,p}(\Omega) \to L^{\frac{dp}{d-p}}(\Omega)$ is continuous, we get compactness for all $1 \le q < \frac{dp}{d-}$ by Hölder interpolation (Proposition 1).

Let A be the unit ball in $W_0^{1,p}(\Omega)$; fix $\delta > 0$, ψ a mollifier, and let $A_\delta \subset C_c^\infty(\mathbb{R}^d)$ be its corresponding regularisation $\{u_\delta | u \in A\}$ (extend u outside Ω by 0). First we show that A_δ is uniformly close to A in $L^1(\Omega)$.

$$\begin{aligned} \|u - u_\delta\|_{1;\Omega} &= \int\limits_\Omega \left| \int\limits_{\mathbb{R}^d} \psi(z)(u(x) - u(x - \delta z)) \, \mathrm{d}z \right| \mathrm{d}x \\ &\le \sup_{|z| \le \delta} \|u(\cdot - z) - u\|_{1;\mathbb{R}^d} \end{aligned}$$

using the same argument as in the proof of Lemma 6. But now we observe that using the difference quotient techniques of Proposition 16, we have that

$$\sup_{|z|\le\delta} \|u(\cdot - z) - u\|_{1;\mathbb{R}^d} \le \sup_{h<\delta} h\|\Delta^h u\|_{1;\mathbb{R}^d} \le \delta \|Du\|_{1;\Omega} \ .$$

[1] I thank Denis Serre for this example.

The last term on the right hand side is bounded by $\delta|\Omega|^{1-1/p}$ from Proposition 1 and that A is the unit ball in $W_0^{1,p}(\Omega)$. Therefore it suffices to show that A_δ is precompact for every $\delta > 0$.

To do so we note that

$$|u_\delta(x)| \leq \delta^{-d} \int\limits_{B_\delta(x)} \psi(\frac{x-y}{\delta})|u(y)| \,\mathrm{d}y \leq \delta^{-d} \sup\psi \|u\|_{1;\Omega}$$

and

$$|\partial u_\delta(x)| \leq \delta^{-d-1} \int\limits_{B_\delta(x)} \left|(\partial\psi)(\frac{x-y}{\delta})u(y)\right| \,\mathrm{d}y \leq \delta^{-d-1} \sup|\partial\psi| \|u\|_{1;\Omega} \,.$$

Therefore A_δ is a bounded, equicontinuous subset of $C(\bar{\Omega})$, and thus precompact in $C(\bar{\Omega})$ by Arzelà-Ascoli Theorem. Therefore A_δ is precompact in $L^1(\Omega)$. □

Exercise 13. Show that the imbedding of $W_0^{1,p}(\Omega) \to C^l(\Omega)$ for $0 \leq l < 1 - \frac{d}{p}$ of Corollary 24 is compact. (Hint: it suffices to show that the image of the unit ball is equicontinuous; this follows from the fact that in Theorem 22 there is a $|\Omega|$ term.)

2.5. **Trace theorems.** In the study of partial differential equations, we often need to consider the boundary value problem, where the restriction of a function u in our Sobolev space to $\partial\Omega$ is required to take certain prescribed values. In the case where u is classical, this restriction operation is not a problem. But in the case where u is a measurable function, as the boundary of an open set has measure 0, one can always freely modify u on that boundary. In this section we try to make sense of this *trace* of a measurable function onto a positive codimension submanifold.

The main trace theorem is a combination of the Gagliardo-Nirenberg-Sobolev Theorem 18 and the Morrey-Sobolev Theorem 22.

Theorem 31. *For $u \in C_c^\infty(\mathbb{R}^d)$, consider its restriction $u^{(M)}$ onto a co-dimension n hyperplane M for $0 \leq n \leq d$. Then there exists some constant $C = C(p,q,k,d,n)$ such that*

$$\|u^{(M)}\|_{q;M} \leq C\|u\|_{p,k;\mathbb{R}^d} \tag{22}$$

whenever

$$k > d\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{n}{q} \,. \tag{23}$$

Proof. Write $\mathbb{R}^d = \mathbb{R}^n_x \times \mathbb{R}^m_y$, where the subscripts denote the variables we will use. In other words, we write $u \in C_c^\infty(\mathbb{R}^d) = u(x,y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Consider the following $C_c^\infty(\mathbb{R}^n_x)$ function

$$v(x) = \|u(x,\cdot)\|_{q;\mathbb{R}^m_y} \,.$$

By the Gagliardo-Nirenberg-Sobolev inequality we have that

$$v(x) \leq C\|u(x,\cdot)\|_{p;\mathbb{R}^m_y}^{1-\frac{m}{k}(\frac{1}{p}-\frac{1}{q})} \|\partial_y^k u(x,\cdot)\|_{p;\mathbb{R}^m_y}^{\frac{m}{k}(\frac{1}{p}-\frac{1}{q})} \leq C\|u(x,\cdot)\|_{p,k;\mathbb{R}^m_y}$$

for $p \leq q \leq \frac{mp}{m-kp}$.

Similarly we use Corollary 23 of the Morrey-Sobolev inequality on x, and have

$$|v(x)| \leq C\|v\|_{p,l;\mathbb{R}^n_x}$$

if $lp > n$. Chaining the two estimates together and differentiating under the integral sign, we have that

$$\sup_{\in\mathbb{R}^n_x} \|u\|_{q;\mathbb{R}^m_y} \leq \tilde{C}\|u\|_{p,j;\mathbb{R}^d}$$

whenever

$$j > d(\frac{1}{p} - \frac{1}{q}) + \frac{n}{q} \ .$$

□

By a density argument this implies the continuous imbedding $W^{k,p}(\mathbb{R}^d) \to L^q(M)$, and similarly $W_0^{k,p}(\Omega) \to L^q(M \cap \Omega)$, for almost every hyperplane M. One can similarly generalize this to traces where M can be any compact smooth submanifold of $\mathbb{R}^d$; it suffices to use a partition of unity argument, and the fact that M can be covered by finitely many balls and in each ball there exist a diffeomorphism with bounded first k derivatives bringing M to a hyperplane. We leave such natural generalisations to the reader.

Notice that if we set the Lebesgue integrability the same on both sides ($p = q$), we see that for each drop in dimension one trades "a little more than" $1/p$ degree of differentiability. This "little more" can be removed in some cases. We give an example below whose proof is based on the Fourier isometry of L^2.

Theorem 32. *Let $u \in C_0^\infty(\mathbb{R}^d)$, and $u^{(M)}$ again, its trace onto a co-dimension n hyperplane. Then if $j > 0$, we have*

$$\|u^{(M)}\|_{2,j;M} \leq C\|u\|_{2,j+\frac{n}{2};\mathbb{R}^d} \ .$$

Proof. By rotation and translation, we can assume that M is the plane $x_1, \ldots, x_n = 0$. Write $\hat{u}$ for the full Fourier transform of u on $\mathbb{R}^d$, and $\tilde{u}$ for the Fourier transform of $u^{(M)}$ on M. The Fourier inversion formula gives

$$\tilde{u}(\xi_{n+1}, \ldots, \xi_d) = \int\limits_{\xi_1,\ldots,\xi_n} \hat{u}(\xi_1, \ldots, \xi_d)\, \mathrm{d}\xi \ .$$

We use the short hand $\eta = (\xi_1, \ldots, \xi_n)$ and $\zeta = (\xi_{n+1}, \ldots, \xi_d)$. So

$$\begin{aligned}
\|u^{(M)}\|^2_{2,j;M} &= \int (1 + |\zeta|^2)^j |\tilde{u}(\zeta)|^2 \,\mathrm{d}\zeta \\
&= \int (1 + |\zeta|^2)^j \left| \int \hat{u}(\eta, \zeta)\, \mathrm{d}\eta \right|^2 \mathrm{d}\zeta \\
&= \int (1 + |\zeta|^2)^j \left(\int (1 + |\eta + \zeta|^2)^{j+\frac{n}{2}} |\hat{u}|^2 \,\mathrm{d}\eta \right) \left(\int (1 + |\eta + \zeta|^2)^{-j-\frac{n}{2}} \,\mathrm{d}\eta \right) \mathrm{d}\zeta
\end{aligned}$$

Consider the last term in the right hand side. Writing $t^2 = 1 + |\zeta|^2$ and $r^2 = |\eta|^2/t^2$, we have

$$\begin{aligned}
\int (1 + |\eta + \zeta|^2)^{-j-\frac{n}{2}} \,\mathrm{d}\eta &= \int (1 + |\eta|^2 + |\zeta|^2)^{-j-\frac{n}{2}} \,\mathrm{d}\eta \\
&= t^{-2j} \int (1 + r^2)^{-j-\frac{n}{2}} \,\mathrm{d}\omega^{n-1} r^{n-1} \,\mathrm{d}r \\
&= ct^{-2j}
\end{aligned}$$

where $c = c(n, j)$ is finite whenever $j > 0$. Plugging this back in we have directly

$$\|u^{(M)}\|_{2,j;M}^2 \leq c \iint (1 + |\eta + \zeta|^2)^{j+\frac{n}{2}} |\hat{u}|^2 \, \mathrm{d}\eta \, \mathrm{d}\zeta = c\|u\|_{2,j+\frac{n}{2};\mathbb{R}^d} \ .$$

$\square$

Examining the proof, we see that the failure of the embedding in the case $j = 0$ is, again, given by a logarithmic divergence. Thus we have the continuous embedding

$$H^k(\mathbb{R}^d) \to H^j(\mathbb{R}^{d-n})$$

when $k - j \geq \frac{n}{2}$ and $j \geq 0$ *except* the case where both equalities are satisfied.

2.6. **Numerology.** In this last section we re-visit the scaling property and use it to obtain a quick heuristic for checking whether a proposed inequality is reasonable. Consider the function $u : \mathbb{R}^d \to \mathbb{R}$ as a map between two vector spaces. The scaling symmetry of a vector space induces two types of scaling on u. The first is the magnification of the source vector space

$$u(x) \mapsto u(\lambda x)$$

and the second is the amplification of the target vector space

$$u(x) \mapsto \kappa u(x) \ .$$

In the following, the phrase "v is a scaling of u by factor (λ, κ)" will mean the replacement $v(x) = \kappa u(\lambda x)$.

Each homogeneous Sobolev space $\mathring{W}^{k,p}(\mathbb{R}^d)$ has, associated to it, a *natural scaling law* $L = L(\lambda, \kappa)$, which we define as

Definition 33. The *natural scaling law* $L = L(\lambda, \kappa)$ associated to $\mathring{W}^{k,p}(\mathbb{R}^d)$ is the function satisfying

$$\|D^k v\|_p = L\|D^k u\|_p$$

whenever v is a (λ, κ)-scaling of u.

For a fixed k, p, d, the set $\{(\lambda, \kappa) \in \mathbb{R}_+^2 | L(\lambda, \kappa) = 1\}$ is called the set of invariant scalings of $\mathring{W}^{k,p}(\mathbb{R}^d)$. This notion of the scaling law and the invariant scalings can be extended to other homogeneous function spaces.

We can compute explicitly what L is for Sobolev spaces. First

$$D^k v(x) = \kappa \lambda^k D^k u(\lambda x) \ .$$

So taking the p^{th} power and integrating,

$$\int |D^k v|^p \, \mathrm{d}x = \kappa^p \lambda^{kp} \int |D^k u|^p \lambda^{-d} \, \mathrm{d}y \ .$$

So the scaling law is

$$L^{(k,p,d)}(\lambda, \kappa) = \kappa \lambda^{k - \frac{d}{p}} \ .$$

Now, how do we use this scaling law? Suppose we are asked to check whether one Sobolev space embeds continuously into another, which boils down to checking whether an inequality of the form

$$\|D^{k_2} u\|_{p_2;\mathbb{R}^{d_2}} \leq C\|D^{k_1} u\|_{p_1;\mathbb{R}^{d_1}}$$

is plausible with some constant depending only on $C = C(k_1, k_2, p_1, p_2, d_1, d_2)$. What we do is consider the (λ,κ) scaling v of u: if this inequality holds for u, then by definition

$$\frac{1}{L^{(k_2,p_2,d_2)}}\|D^{k_2}v\|_{p_2;d_2} \le \frac{C}{L^{(k_1,p_1,d_1)}}\|D^{k_1}v\|_{p_1;d_1} .$$

Now consider the invariant scaling set for k_2, p_2, d_2. Solving for κ we get $\kappa = \lambda^{\frac{d_2}{p_2}-k_2}$. Then the restriction of $L^{(k_1,p_1,d_1)}$ to this set becomes

$$L = \lambda^{\frac{d_2}{p_2}-k_2+k_1-\frac{d_1}{p_1}} = \lambda^c .$$

So if $c \neq 0$, by choosing a scaling corresponding to either a really large or really small λ, we can make the right hand side arbitrarily small, while fixing the left hand side to be constant size, and thus obtaining a contradiction. Hence *a necessary condition for a embedding of one homogeneous Sobolev space into another is the equality of their scaling laws*.

Let us test this on the trace theorem of the previous section. Then $d_1 = d$, $d_2 = d - n$, $k_1 = k$, $p_1 = p$, $k_2 = 0$, $p_2 = q$. Then a homogeneous embedding will require

$$\frac{d-n}{q} + k - \frac{d}{p} = 0$$

or

$$k = d(\frac{1}{p} - \frac{1}{q}) + \frac{n}{q}$$

precisely the logarithmically divergent end-point case which we disallow. (Notice that insofar as scaling laws are concerned, $\log(x) \sim x^0$.)

For another example we can look at the interpolated Sobolev inequality. Suppose we want

$$\|u\|_{q;\mathbb{R}^d} \le C\|u\|^{1-t}_{p;\mathbb{R}^d}\|\partial^k u\|^t_{p;\mathbb{R}^d}$$

the scaling law of the left hand side is invariant when $\kappa = \lambda^{d/q}$. The right hand side has scaling by

$$(L^{(0,p,d)})^{1-t}(L^{(k,p,d)})^t = \kappa\lambda^{-(1-t)\frac{d}{p}}\lambda^{t(k-\frac{d}{p})}$$

so scaling invariance requires

$$\frac{d}{q} + tk - \frac{d}{p} = 0$$

which tells us that the interpolation exponent must be

$$0 \le t = \frac{d}{k}(\frac{1}{p} - \frac{1}{q}) \le 1$$

(which was used in the proof of Theorem 31), and gives us precisely the range of admissible Sobolev inequalities (as long as $q < \infty$).

This last example here also demonstrates how one is to treat embedding inequalities with non-homogeneous Sobolev spaces: instead of considering $\|u\|_{p,k;\mathbb{R}^d}$, we should consider the interpolation $\|u\|^{1-t}_{p;\mathbb{R}^d}\|\partial^k u\|^t_{p;\mathbb{R}^d}$ which will have a homogeneous scaling law, while also be roughly equivalent to the inhomogeneous Sobolev norm.

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