HOW TO DERIVE THE KERR METRIC BY CHEATING QUITE A BIT; A LECTURE GIVEN TO MAT 451 ON APRIL 23, 2009

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ABSTRACT. In this set of notes, an unconventional method of deriving the Kerr metric is presented.

1. INTRODUCTION AND THE FIRST ANSATZ

In this note we give a heuristic derivation of the Kerr metric, in a way quite significantly different from the classical methods. This is in no way a formal write-up, so for a more rigorous derivation, and for references, please see the wonderful article by Roberto Bergamini and Stefano Viaggiu, "A novel derivation for Kerr metric in Papapetrou gauge."[1] The method described herein is inspired by Marc Mars' paper "A spacetime characterization of the Kerr metric"[2], and also by the author's 2009 PhD dissertation.

The question we seek to answer is: *find a solution of the Einstein vacuum equations that is stationary and axially symmetric.* However, to actually answer the question, we will need to impose very significant, and not *ab initio* justifiable, constraints. It is only with great hindsight (that we know the solution we seek already) that the constraints seem natural. On the other hand, we will try to argue that these constraints are not completely wild guesses: by following a particular coherent chain of thought, it may have been possible to have obtained the Kerr metric through this method with no prior knowledge of the metric.

The principal argument here is thus: we first consider the Ernst potential on a stationary solution to the Einstein equations. Next we compute the Ernst potential, the Ernst two-form, and the Weyl curvature of the Schwarzschild metric. By observing that Schwarzschild is algebraically special, we make the ansatz that we want to search for a similarly algebraically special solution. We find that through some cosmic coïncidences, there exists additional solutions beside the Schwarzschild metric that satisfy the algebraic condition. And through a calculation we (almost) show that this is the Kerr metric.

First we begin by quantifying the type of solutions we are actually looking for. The presentation here is standard, and the assumptions given here are made from first principles and form the common starting point of every "derivation" of the Kerr metric. In particular, we define what a solution is and what the symmetry conditions means. We seek to answer the following problem

Problem 1. *We want to find a four-dimensional Lorentzian manifold (M, g_{ab}) such that it is Ricci flat. We ask that it admits two Killing vector fields τ^a, η^a which*

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[1]*Class. Quantum Grav.* 21 4567–4573 (2004)

[2]*Class. Quantum Grav.* 16 2507–2523 (1999)

commute ($[\tau, \eta] = 0$) and whose normal distribution $\{\tau, \eta\}^{\perp} \subset TM$ is space-like and integrable.

It is clear that from the formulation above, our solution manifold can be ruled by two transverse foliations: one by the integrated normal distribution, one by the tangential distribution to τ, η. Each of the foliation is two dimensional, so the induced metric on it can be diagonalized (at least locally). Therefore immediately the formulation of the problem admits the following ansatz

$$(1) \qquad ds^2 = -Sdt^2 + 2Qdtd\phi + Rd\phi^2 + Vdr^2 + Wd\theta^2$$

where $\partial_t = \tau$ and $\partial_\phi = \eta$ are the Killing vector fields and the functions S, Q, R, V, W are functions of r and θ only. It suffices to solve for the five unknown functions and show that the orbits of η are closed.

Unfortunately, if one writes down the Ricci-flat condition for the above ansatz, one gets a monstrous set of equations that takes tens of pages to be written.

A quick note: in the following, for a tensor quantity $X_{abc\cdots d}$, we write X^2 for the full contraction against itself

$$X^2 = X_{abc\cdots d}X_{lmn\cdots o}g^{al}g^{bm}g^{cn}\cdots g^{do} \ .$$

In other words, we write X^2 for $g(X, X)$ where $g(\cdot, \cdot)$ is, by an abuse of notation, the inner product induced on the tensor bundle by the Lorentzian metric. In particular, X^2 is a scalar that can be of arbitrary sign.

2. THE ALGEBRAIC ALIGNMENT CONDITION

To further simplify the equations, we want to impose additional constraints. Here we give a line of (perhaps questionable) reasoning that leads to certain algebraic constraints.

2.1. The Ernst two-form and Ernst potential.
Consider the Killing vector field τ^a. We write τ_a for its dual one-form. Killing's equation implies that $\nabla_a \tau_b$ is anti-symmetric, so the two form

$$(2) \qquad F_{ab} = (d\tau)_{ab} = \nabla_a \tau_b - \nabla_b \tau_a = 2\nabla_a \tau_b$$

is defined. This two-form is called the *Ernst two-form*. As is well known, the second covariant derivatives of a Killing vector field is given by the Riemann curvature tensor

$$(3) \qquad \nabla_c F_{ab} = 2\nabla_c \nabla_a \tau_b = 2R_{dcab}\tau^d \ .$$

This implies that F_{ab} is a Maxwell field with source:

$$(4a) \qquad\qquad\qquad \nabla_{[c}F_{ab]} = 0$$

$$(4b) \qquad\qquad\qquad \nabla^a F_{ab} = -2R_{db}\tau^d$$

where the first line is due to either the exterior algebra $d \circ d = 0$ or the first Bianchi identity on R_{abcd} (the two are equivalent), and the second one by contracting (3) above. Observe that for a solution of the Einstein vacuum equation, F_{ab} is a free Maxwell field. From now on we will make the assumption that we are considering a solution of the Einstein vacuum equation[3].

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[3]Some of the computations below cannot be done in the non-vacuum case. I'll leave it to the readers to figure out where the difficulties are.

As a Maxwell field, the form F_{ab} has a natural electromagnetic decomposition

$$(5) \qquad E_a := F_{ab}\tau^b = 2(\nabla_a\tau_b)\tau^b = \nabla_a\tau^2 \ , \quad B_a = (^*F)_{ab}\tau^b$$

where $*$ is the Hodge dual operator, which we can write in coördinates

$$(6) \qquad (^*F)_{ab} = \frac{1}{2}\epsilon_{abcd}F^{cd}$$

where ϵ_{abcd} is the volume form (or the Levi-Civita symbol in an orthonormal frame). When τ^a is not a null-vector, we claim that, as in the case of standard Maxwell theory on Minkowski space (where τ is replaced by the time coördinate derivative), we can reconstitute F_{ab} from the electromagnetic components using the following algebraic identity

$$(7) \qquad \tau^2 F_{ab} = E_a\tau_b - E_b\tau_a - \epsilon_{abcd}B^c\tau^d$$

Now, let us examine the magnetic part B_a. It is also called the *twist* of the Killing vector field τ^a. Observe that by Frobenius' theorem, the normal bundle to the vector field τ^a is integrable if and only if $B_a = 0$. In other words, the twist tells us whether τ^a is hypersurface-orthogonal. As seen above, E_a is exact: it arises from the potential τ^2. We claim that since τ is Killing, and since that the space is Ricci-flat, B_a also has a potential. Observe that $d^*F = -^*\delta^*F = 0$. So $^*F$ is a closed two-form. Now use the Cartan relation

$$\mathcal{L}_X\omega = d(i_X\omega) + i_X(d\omega)$$

where ω is a form and X a vector field by taking $X = \tau$ and $\omega = {}^*F$. Since τ is Killing, and F is geometric, we must have that $\mathcal{L}_\tau {}^*F = 0$. Therefore we conclude that

$$(8) \qquad d(i_\tau {}^*F) = 0 = dB \ .$$

Now if we assume our space-time is simply connected (or let's say we look at a simply connected domain), we can define, up to a constant, a real-valued scalar Θ such that $d\Theta = B$. This Θ is called the *Ernst potential*

So why do we care about the Ernst two-form? Recall our ansatz (1), we know that $\tau^2 = -S$. Now, observe that the dual one-form to the Killing vector field τ is given by

$$\tau^\flat = -Sdt + Qd\phi$$

and so using that S and Q are independent of t and ϕ, we have

$$(9) \qquad F = \partial_r Sdt \wedge dr + \partial_\theta Sdt \wedge d\theta - \partial_r Qd\phi \wedge dr - \partial_\theta Qd\phi \wedge d\theta \ .$$

On the other hand, taking the orientation

$$(10) \qquad dvol = \left((SR + Q^2)VW\right)^{\frac{1}{2}} dt \wedge d\phi \wedge dr \wedge d\theta$$

we can compute using (7) an expression for F in terms of E and B. Observe that τ^2 and Θ should both be independent of t and ϕ, so that

$$B = \partial_r\Theta dr + \partial_\theta\Theta d\theta \ , \quad E = -\partial_r Sdr - \partial_\theta Sd\theta \ .$$

Therefore

$$(11a) \quad E \wedge \tau^\flat = -S\partial_r Sdt \wedge dr - S\partial_\theta Sdt \wedge d\theta + Q\partial_r Sd\phi \wedge dr + Q\partial_\theta Sd\phi \wedge d\theta$$

$$(11b) \quad B \wedge \tau^\flat = S\partial_r\Theta dt \wedge dr + S\partial_\theta\Theta dt \wedge d\theta - Q\partial_r\Theta d\phi \wedge dr - Q\partial_\theta\Theta d\phi \wedge d\theta$$

and so, by observing that the inverse metric is given by

$$-\frac{R}{SR+Q^2}(\partial_t)^2 + \frac{2Q}{SR+Q^2}\partial_t\partial_\phi + \frac{S}{SR+Q^2}(\partial_\phi)^2 + \frac{1}{V}(\partial_r)^2 + \frac{1}{W}(\partial_\theta)^2$$

we can write

$$
\begin{aligned}
(12)\qquad -SF = &-S\partial_r S dt \wedge dr - S\partial_\theta S dt \wedge d\theta + Q\partial_r S d\phi \wedge dr + Q\partial_\theta S d\phi \wedge d\theta \\
&- \left((SR+Q^2)VW\right)^{\frac{1}{2}} \left(\frac{\partial_r\Theta}{V} d\phi \wedge d\theta - \frac{\partial_\theta\Theta}{W} d\phi \wedge dr\right)
\end{aligned}
$$

which implies that

$$(13a)\qquad\qquad S\partial_r Q = Q\partial_r S + \sqrt{(SR+Q^2)VW}\frac{\partial_\theta\Theta}{W}$$

$$(13b)\qquad\qquad S\partial_\theta Q = Q\partial_\theta S - \sqrt{(SR+Q^2)VW}\frac{\partial_r\Theta}{V}$$

2.2. The anti-self-dual fields and complexification.
For ease of algebraic manipulations, often we consider the anti-self-dual versions of two-forms. Observe that on a four-dimensional Lorentzian manifold, the Hodge star operator takes two-forms to two-forms, and squares to -1. This implies that its eigenvalues can only be $\pm i$. So we complexify our geometry by $\otimes_{\mathbb{R}}\mathbb{C}$ linearly (so in particular

$$(X + iY)^2 = X^2 + 2ig(X,Y) - Y^2$$

and not the Hermitian product). It is clear that (via a little bit of linear algebra) that the space of two-forms $\Lambda^2 T^*M$ splits after complexification

$$(14)\qquad\qquad \Lambda^2 T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda_+ \oplus \Lambda_-$$

where $\Lambda_\pm$ are spaces of complex-valued two-forms that have eigenvalues $\pm i$ under $*$ respectively. It is also clear that there is a natural isomorphism from $\Lambda^2 T^*M$ to each of $\Lambda_\pm$ (they all have real dimension 6).

So instead of focusing on real-valued two-forms, we'll focus on Λ_-, elements of which are called *anti-self-dual two-forms*. The canonical isomorphism between $\Lambda^2 T^*M \leftrightarrow \Lambda_-$ is given by

$$\Lambda^2 T^*M \ni X_{ab} \leftrightarrow \frac{1}{2}(X_{ab} + i^* X_{ab}) = \mathcal{X}_{ab} \in \Lambda_- \ .$$

The anti-self-dual forms enjoy many marvelous algebraic properties, especially with regards to tensor products and their traces. For a list of such properties, see the paper of Mars referenced before or Chapter 2 of the author's PhD dissertation. Those algebraic properties, however, will not need to be used in this note.

In the following we denote by

$$(15)\qquad\qquad \mathcal{F}_{ab} := \frac{1}{2}(F_{ab} + i^* F_{ab})$$

the anti-self-dual Ernst two-form. Observe that it is a closed two-form, and thus $\mathcal{F}_{ab}\tau^a$ is also closed by the argument before, and on a simply-connected domain is

given by the potential σ:

(16a)
$$\sigma = \frac{S-1}{2} - i\frac{\Theta}{2}$$

(16b)
$$\nabla_b \sigma = \nabla_b(\frac{S}{2} - i\frac{\Theta}{2})$$
$$= -\frac{1}{2}(E_b + iB_b) = \mathcal{F}_{ab}\tau^a \ .$$

We call σ the complex Ernst potential. (The normalization that $\Re\sigma = (S-1)/2$ is to accommodate the physical assumption of asymptotic flatness: near "spatial infinity", τ^a is expected to approach a time translation, meaning that $\tau^2 \to -1$. The chosen normalization allows $\Re\sigma \to 0$ at spatial infinity as a consequence.)

Next, one can observe that the Riemann curvature tensor can be viewed as a symmetric map from $\Lambda^2 T^* M$ to itself. The Ricci decomposition of the Riemann curvature tensor into

$$\text{Riemann} = \text{Weyl} \oplus \text{traceless Ricci} \oplus \text{Scalar}$$

is a purely algebraic decomposition on the space of such maps (see the handout for week 2 of this class for more information). The property we will use is that for the Weyl conformal tensor, we can define its left and right Hodge duals

$$^*W_{abcd} = \frac{1}{2}\epsilon_{ab}{}^{ef}W_{efcd} \ ; \quad W^*_{abcd} = \frac{1}{2}W_{abef}\epsilon^{ef}{}_{cd}$$

and verify that

$$^*W_{abcd} = W^*_{abcd}$$

which is equivalent to the statement that, viewing the Weyl curvature as a map from two-forms to two-forms, it commutes with the Hodge star operator. In any case, since the Hodge dual is well-defined (the left and right actions are equal), we can define the anti-self-dual Weyl curvature as

(17)
$$\mathcal{C}_{abcd} := \frac{1}{2}(W_{abcd} + i^*W_{abcd}) \ .$$

2.3. **Principal null directions.** Let X_{ab} be a real-valued two-form on our four dimensional Lorentzian manifold. Consider the eigenvalue problem for X_{ab}. In the Riemannian case, because the metric is positive definite, there exists no nontrivial solutions to

$$X_{ab}r^b = \lambda r_a$$

since by contracting against the vector r^a, we obtain

$$0 = r^a X_{ab}r^b = \lambda g(r, r)$$

where the first equality follows from the anti-symmetry of two-forms. So either r is the zero-vector or that it is in the kernel of X. Contrast to the case where the metric is pseudo-Riemannian. The expression above tells us that either r is a null vector, or it has eigenvalue 0.

We say that a null vector r^a is a principal null vector of the two-form X_{ab} if it is an eigenvector. The eigenvalue equation can be evidently re-written in the following form

(18)
$$r_{[c}X_{a]b}r^b = 0 \ .$$

Now treating the Weyl curvature as a symmetric map from two-forms to two-forms, we can also ask for the eigenvalues and eigenvectors. Naturally the eigenvectors and eigenvalues of the two-forms can now be lifted to the level of the Weyl curvature, and thus we say that a null vector r^a is a principal null vector of the Weyl curvature tensor W_{abcd} if

$$(19) \qquad\qquad r^b r_{[e} W_{a]bc[d} r_{f]} r^c = 0 \ .$$

(Observe that, in form, (19) is a simple generalization of (18).) Since r^a is a null vector, it makes no sense to try to normalize it to unit length, so we can't find a preferred unit eigenvector. Hence it is traditional also to refer to principal null directions instead of the principal null vectors. (Furthermore, the space of null directions form a $\mathbb{S}^2$ bundle over the manifold, with a conformal structure induced by the Lorentz transformations [i.e. Local diffeomorphisms]. So working, at least locally, with elements in the space of null directions can be reduced to working on $\mathbb{CP}^1$, where a lot of algebraic tools are available. This is sort of one way to look at spinors in the 4-dimensional, Lorentzian case.)

Following is a theorem about the existence of principal null directions for two-forms and Weyl-fields (resp. spin 1 and spin 2 fields). The two-form case is classical and well-known in the physics literature. The Weyl-field case is due to Petrov.

Theorem 2. *Let X_{ab} be a real-valued two-form, and W_{abcd} be a $(0,4)$-tensor satisfying all algebraic symmetries of the Weyl conformal curvature, on a four dimensional Lorentzian manifold. Then at every point p, X_{ab} has two (possibly coïncidental) principal null directions, and W_{abcd} has four (possibly coïncidental) principal null directions, unless the tensors vanish identically.*

From the spinor point of view, the above theorem is simple to prove[4]. A quick sketch: as we remarked above that the space of null-directions can be identified with $\mathbb{CP}^1$. A (perhaps not-so-)simple calculation verifies that under this identification (18) and (19) become a degree-two and a degree-four polynomial on $\mathbb{CP}^1$ respectively. By the fundamental theorem of calculus, the polynomials have two and four zeros respectively when counted with multiplicity.

We say X_{ab} is *non-degenerate* or *non-null* if the two principal null directions are distinct. Using a bit of linear algebra, one sees that this is equivalent to its anti-self-dual part having non-zero norm

$$\mathcal{X}^2 = \mathcal{X}_{ab} \mathcal{X}^{ab} \neq 0 \ .$$

Let l^a and k^a stand for future pointing vector-fields corresponding to the two distinct principal null directions, we can ask that they are normalized to have $l^a k_a = -1$. Then we have the re-consitution formula for a non-null two-form:

$$(20) \qquad\qquad \mathcal{X}_{ab} = \frac{1}{2}(-\mathcal{X}^2)^{1/2}(l_a k_b - k_a l_b + i\epsilon_{abcd} l^c k^d)$$

where the square root is taken with respect to complex numbers, so there exists two roots; therefore up to exchanging the labels l^a and k^a, the above equation is well-defined. From (20) it is also clear that

$$(21) \qquad\qquad \mathcal{X}^2 = -4(\mathcal{X}_{ab} k^a l^b)^2 \ .$$

A similar statement can be had for a special type of Weyl fields. We say a Weyl field is *Petrov type D* if it only has two distinct principal null directions, and each of

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[4]See Penrose and Rindler, *Spinors and space-time* for example.

the directions has algebraic multiplicity 2. Roughly speaking, one can think such a Weyl field as a tensor product $W_{abcd} \sim X_{ab} \otimes X_{cd}$ of non-null two-forms. (A more precise notion of this is given by the concept of a "symmetric spinor product" defined in the author's PhD thesis.) For a Petrov type D field, we can write down an analogous formula to (20), which we will omit here. Analogous to (21), we see that a Petrov type D field W_{abcd} is similarly characterized by its two principal null directions and the scalar $\mathcal{C}_{abcd}k^a l^b k^c l^d$ (or $\mathcal{C}^2$) where $\mathcal{C}_{abcd}$ is the anti-self-dual part of W_{abcd}.

2.4. The Schwarzschild metric.
Let us now examine the Schwarzschild metric

$$(22) \qquad ds^2 = -(1 - \frac{2M}{r})dt^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ .$$

We take $\tau^a = \partial_t$, so $F = \frac{2M}{r^2}dt \wedge dr$. Now notice that $B = 0$, since ∂_t is hypersurface orthogonal. Therefore the complex Ernst potential is given by

$$(23) \qquad \sigma = \frac{S-1}{2} = -\frac{M}{r} \ .$$

Now, a direct computation (which we'll omit here) shows that

$$(24) \qquad \mathcal{F}^2 = -\frac{4M^2}{r^4} = -\frac{4}{M^2}\sigma^4$$

$$(25) \qquad \mathcal{C}^2 = \frac{24M^2}{r^6} = \frac{24}{M^4}\sigma^6 \ .$$

What about the principal null directions? It is easy to see that F_{ab} is non-null and W_{abcd} is type D using spherical symmetry of the Schwarzschild metric. Simply speaking, if r^a is a vector defined at some point p such that it is a principal null direction. Let O be an element of $SO(3)$ such that its action on (M, g) fixes the point p. Then the induced diffeomorphism is a map from T_pM to itself. Consider the vector $(O^*r)^a$, it will necessarily be another principal null direction. Using the algebraic classification theorem, one sees now that r^a must be fixed by the action of O. Therefore any principal null direction must either in the plane spanned by ∂_t and ∂_r. Since W_{abcd} and F_{ab} do not vanish identically (else we are in Minkowski space), the only possible vectors which can be the principal null directions are (after normalizing to $k^a l_a = -1$ and requiring them to be future pointing)

$$(26) \qquad k = \frac{1}{\sqrt{1 - \frac{2M}{r}}}\partial_t + \sqrt{1 - \frac{2M}{r}}\partial_r \ , \qquad l = \frac{1}{\sqrt{1 - \frac{2M}{r}}}\partial_t - \sqrt{1 - \frac{2M}{r}}\partial_r \ .$$

To see that both k and l are principal null directions, we use the fact that the Schwarzschild metric also has a discrete time-reflection symmetry sending $t \leftrightarrow -t$. Under this change $k \leftrightarrow -l$. Hence the algebraic multiplicity of k and l as principal null directions must be equal. Therefore F_{ab} is non-null and W_{abcd} is type D.

2.5. Revised ansatz.
In view of the special algebraic properties of the Schwarzschild metric, we revise our initial guess and ask for a solution to the following problem

Problem 3. *Find a four-dimensional Lorentzian manifold (M, g_{ab}) such that the following conditions hold:*

(1) *(M, g_{ab}) is Ricci flat.*
(2) *It admits two Killing vector fields τ^a, η^a which commute ($[\tau, \eta] = 0$) and whose normal distribution is space-like and integrable.*

(3) *The Ernst two-form is non-null; the Weyl curvature has Petrov type D; and their principal null directions are aligned.*

(4) *The Ernst potential satisfies the following: there exists a real valued constant M such that*

$$\mathcal{F}^2 = -\frac{4}{M^2}\sigma^4 \ , \qquad \mathcal{C}^2 = \frac{24}{M^4}\sigma^6 \ .$$

(Recall that the real Ernst potential Θ is defined only up to a (real) constant. The condition here should read to mean that there exists a normalization for Θ such that the conditions described here holds. One sees that the only possible normalization in the asymptotic flat case is by assuming Θ vanishes at spatial infinity.)

In general, of course, it is not immediately obvious that a solution to Einstein's equation with all the listed properties exist. In practice, it suffices to try to calculate under a contradiction is found, or until a self-consistent answer emerges. Here, we use our amazing hindsight that the Kerr-metric actually satisfy the above conditions in the formulation of the above problem. Of course, we claim that the *asking* of the above question is not completely unreasonable in view of the properties of the Schwarzschild metric.

3. DERIVING THE KERR METRIC

In this section we show how the Kerr metric may be (in a large part) derived by studying Problem 3.

The main result that we rely on is a lemma given in Mars' 1999 paper[5].

Lemma 4. *We can define the real-valued function y and z by*

$$-\sigma^{-1} = y + iz \ .$$

Then there exists a non-negative real number B such that $B > z^2$, and

$$(27a) \qquad\qquad\qquad (\nabla y)^2 = \frac{y^2 - 2y + B}{M^2(y^2 + z^2)}$$

$$(27b) \qquad\qquad\qquad (\nabla z)^2 = \frac{B - z^2}{M^2(y^2 + z^2)} \ .$$

The proof is omitted here. I'll give a basic sketch of the idea. Write k and l for the two mutually-normalized vector fields corresponding to the principal null directions. Consider the integral curves of k (or l respectively). By the Goldberg-Sachs theorem the congruences defined by the family of integral curves are geodesic and shear free. Observe also that the only components of $\mathcal{C}_{abcd}$ comes from the $\mathcal{C}(k,l,k,l)$ component and others that can be related to it using the algebraic symmetries of Weyl fields. Now consider the second Bianchi identity applied to $\mathcal{C}_{abcd}$ (it is here we use the Ricci-flat condition: that the Weyl field obeys the second Bianchi identity), one sees that this implies

$$\epsilon_{abcd}l^a k^b \nabla^c y = 0 \ , \quad l^a \nabla_a z = k^a \nabla_a z = 0 \ ,$$

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[5]If you actually look at the paper, you'd see that there are some factors of 2 differences in a lot of the statements. These are related to the fact that our definitions of anti-self-dual forms differs by a factor of 2, and that our definitions of the Ernst two-form and the Ernst potential also differ by a factor of 2.

which, in particular, shows that ∇z is space-like. A rather complicated calculation then shows that

$$M^2(y^2 + z^2)(\nabla z)^2 + z^2$$

is constant, using the equation induced on σ from the second Bianchi identity. Using that ∇z is space-like, we have that the constant B must be greater than or equal to z^2. The statement from $(\nabla y)^2$ follows from the following observation:

$$(\nabla \sigma)^2 = \mathcal{F}_{ab}\tau^a \mathcal{F}^{cb}\tau_c = \frac{1}{4}\mathcal{F}^2\tau^2 = -\frac{1}{M^2}\sigma^4\tau^2 .$$

So

$$(\nabla \frac{1}{\sigma})^2 = -\frac{\tau^2}{M^2} = \frac{1 + 2\Re\sigma}{M^2} .$$

By simple algebraic manipulations from the definition of y and z, we obtain the statement on $(\nabla y)^2$.

By the above lemma and its proof, we see that at points where $B \neq z^2$ and $y^2 - y + B \neq 0$, y and z are independent, non-degenerate scalar functions. Furthermore, as they are geometric quantities defined from objects that are symmetric under the τ and η actions, we must have $\tau(y) = \tau(z) = \eta(y) = \eta(z) = 0$. So we can take y and z to be coördinate functions on (subsets of) the surface orthogonal to τ, η.

We will make also the following guess: by the form of $(\nabla z)^2$, we can reasonably expect the change of variables $z = \sqrt{B}\cos\theta$ may patch together where $z^2 = B$ and resolve the coördinate singularity. Furthermore we will take $a = M\sqrt{B}$. In addition, let $r = My$. Then we write

$$(28a) \qquad \sigma = -\frac{M}{r + ia\cos\theta} = \frac{-Mr + iMa\cos\theta}{r^2 + a^2\cos^2\theta}$$

$$(28b) \qquad \Rightarrow S = 2\Re\sigma + 1 = 1 - \frac{2Mr}{r^2 + a^2\cos^2\theta}$$

$$(28c) \qquad \Rightarrow \Theta = -2\Im\sigma = -\frac{2Ma\cos\theta}{r^2 + a^2\cos^2\theta}$$

$$(28d) \qquad (\nabla r)^2 = \frac{r^2 - 2Mr + a^2}{r^2 + a^2\cos^2\theta}$$

$$(28e) \qquad (\nabla\theta)^2 = \frac{1}{r^2 + a^2\cos^2\theta}$$

so going back to ansatz (1), we have that

$$(29) \quad ds^2 = -(1 - \frac{2Mr}{r^2 + a^2\cos^2\theta})dt^2 + 2Qdtd\phi + Rd\phi^2$$
$$+ \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 .$$

Now look at (13), we can re-write it as

$$(30) \qquad \frac{\partial_r(Q/S)}{\partial_\theta(Q/S)} = -\frac{V\partial_\theta\Theta}{W\partial_r\Theta} = -\frac{(r^2 - a^2\cos^2\theta)\sin\theta}{2r\cos\theta(r^2 - 2Mr + a^2)} .$$

Observe now that for $K(r, \theta)$ given by

$$(31) \qquad K(r, \theta) := \frac{2Ma\sin\theta}{(r^2 - 2Mr + a^2\cos^2\theta)^2}$$

we have that

$$\frac{\partial}{\partial\theta}[(a^2\cos^2\theta - r^2)\sin\theta K(r,\theta)] = \frac{\partial}{\partial r}[2r\cos\theta(r^2 - 2Mr + a^2)K(r,\theta)]$$

which implies that (with a rather non-trivial computation)

$$\frac{Q}{S} = \frac{2Mar\sin^2\theta}{r^2 - 2Mr + a^2\cos^2\theta} \ .$$

This in turn tells us that

$$Q = \frac{2Mar\sin^2\theta}{r^2 + a^2\cos^2\theta} \ .$$

By examining (13) again, we can solve for R purely algebraically (a computation I'll omit here) to arrive at

$$R = \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{r^2 + a^2\cos^2\theta}\right) \ .$$

This gives us the Kerr metric.

Notice that, however, the computation of Q/S has the freedom to add a constant. This reflects the fact that our computation, in the end, is completely local. In other words, we know that on a local neighborhood the metric takes a given expression, what we don't know is whether the Killing vector field represented by ∂_ϕ actually has closed orbits! The freedom to add a constant factor to Q/S reflects the fact that any constant coefficient linear combination

$$\tilde{\eta} = c_1\tau + c_2\eta$$

is again a Killing vector field which commutes with τ. So our local coördinate form may be chosen initially such that ∂_ϕ coïncides with $\tilde{\eta}$, which does not have closed orbits.

A proper argument to get rid of this degree of freedom requires a careful examination of the properties of the bifurcate event horizon. In particular, assuming the space-time has a bifurcate event horizon, one can easily argue that both τ and η are tangent to the bifurcate sphere, and in fact are multiples of each other. This allows us to fix the unknown constant. This argument is similar to the assumption used by Chandrasekhar in *Mathematical Theory of Blackholes* to analytically obtain the Kerr metric.

Lastly, one may ask about the title of this lecture, in particular about the "cheating quite a bit" part. The algebraic conditions derived in Section 2 are actually not too unreasonable. The primary "cheat" employed here is actually in the integrating factor $K(r,\theta)$ in (31). Whereas with our perfect hindsight, we can easily find the correct integrating factor, it is extremely difficult (insofar as the author's limited computational capabilities is concerned) to find the integrating factor just given (30).

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