DERIVATIVES: A GEOMETRIC SUPPLEMENT

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These short notes grew out of my reflections on my own education in differential geometry and mathematical physics, and represent wistfully how I wish I were taught when I was younger. This is not to say the prevailing approach to the foundations are not useful; to the contrary, a good understanding of the presentation given in standard textbooks (such as that of Kobayashi and Nomizu [KN96a, KN96b]) are indispensable for actually doing geometry. The points of view here may however provide some additional motivation and better conceptual understanding of the how and the why of all those complicated differential geometric notions.

As such, the notes are by no means complete or pedagogical. The author has shamelessly assumed a certain preliminary education and will make no attempt to define certain objects that come into play, relying instead on the aforementioned standard textbooks. Any errors should however be entirely blamed on the author. Perhaps one day these will be properly expanded into some lecture notes on differential geometry. We shall see.

1. Fibred manifolds, bundles, and all that

At the heart of it the motivation for defining various notions of derivatives arises from trying to generalise the notion of *tangent map* between manifolds. Recall that given smooth manifolds M and N and a smooth map $f : M \to N$ its associated tangent map is denoted[1] $\mathsf{T}f : \mathsf{T}M \to \mathsf{T}N$. This tangent map can be identified by requiring that the projection maps $\pi_{M,N}\mathsf{T}M, \mathsf{T}N \to M, N$ commute as

$$\pi_N \circ \mathsf{T}f = f \circ \pi_M \tag{1}$$

and that for scalar function $\phi : N \to \mathbb{R}$ we have that $\mathsf{T}f[X](\phi) = X(f \circ \phi)$ for all elements $X \in \mathsf{T}M$.

In studying differential geometry, however, often we are faced with the task of studying "maps" where the co-domain depends on the domain. That is to say, there is a family of manifolds N_p parametrized by $p \in M$, and we wish to study the function

$$f : M \ni p \mapsto q \in N_p.$$

As it stands, however, the situation is too general: there is no smooth structure describing the transition from one N_p to the next. Hence we glue all the N_p together using the manifold M and study this big object as a manifold. This gives rise to the definition.

Definition 1. A *fibred manifold* is the triple (E, M, π) where E and M are smooth manifolds, and $\pi : E \to M$ is a smooth surjection with surjective tangent map

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[1] Other notations include $\mathrm{d}f$ and f_*.

$\mathsf{T}\pi : \mathsf{T}E \to \mathsf{T}M$. We say that E is the *total manifold*, M is the *base*, and π the projection map. We denote by $E_p \stackrel{\text{def}}{=} \pi^{-1}(p)$ the *fibre* over the point $p \in M$.

Remark 1. Note that when E is finite dimensional, so is M and $\dim(M) \leq \dim(E)$.

Definition 2. A *section* of a fibred manifold (E, M, π) over an open subset $U \subset M$ is defined as a smooth mapping $s_U : U \to E$ such that $\pi \circ s_U = \mathbf{1}_U$. A *global section* is a section over M.

Note that fibred manifolds must admit local sections; this is a consequence of the *submersion theorem* of differential geometry. Global sections however can fail to exist.

exa:noglobalsection **Example 1.** Let $M = \mathbb{S}^1$ and $E = \mathbb{S}^1 \times \mathbb{R}$. And let the projection be

$$\pi(\theta, s) = 2\theta, \quad \theta \in \mathbb{S}^1.$$

It is easy to see that a global section cannot exist.

Observe that the section is a differentiable mapping, hence it has an associated tangent map $\mathsf{T}s_U : \mathsf{T}U \to \mathsf{T}E$ such that the composition $\mathsf{T}\pi \circ \mathsf{T}s_U = \mathbf{1}_{\mathsf{T}U}$. The following terminology should be natural.

Definition 3. A vector X in $\mathsf{T}E$ is said to be *vertical* if $\mathsf{T}\pi[X] = 0$.

One may ask whether there is a natural definition of "horizontal": the answer is no. And in the next section we will see it as the motivation for the definition of the connection. Before we continue however let us recall some the names of some objects which have additional structure beyond that of a simple fibred manifold.

1.1. Bundle structure.

A fibre bundle is simply a fibred manifold where all the fibres look the same. This is the case that more resembles the situation of mappings $f : M \to N$. More precisely:

Definition 4. A *smooth fibre bundle* is the quadruple (E, M, π, N) where E, M, N are smooth manifolds, and the map $\pi : E \to M$ is a submersion (hence making (E, M, π) a fibred manifold). This splitting is "locally trivial", meaning that at every point $p \in M$ there exists an open set $U_p \ni p$ and a diffeomorphism $\phi : U_p \times N \to E$ such that $\phi(p', q) \in E_p$ for $p' \in U_p$ and $q \in N$. The manifold N is said to be the *fibre*.

Going back to the fibred manifold in Example 1, we see that it is a fibre bundle with the fibre being two disjoint copies of $\mathbb{R}$.

Now let U, V be open sets of M such that $U \cap V \neq \emptyset$, and such that we have trivialisations ϕ_U and ϕ_V. In general the trivialisations need not agree: let $p \in U \cap V$ and $q \in N$, the trivialisation could be such that $\phi_U(p, q) \neq \phi_V(p, q)$. However, by definition we know that both $\phi_U|_{p \times N}$ and $\phi_V|_{p \times N}$ define diffeomorphisms with the fibre, so necessarily there exists a diffeomorphism $\psi_{VU} : (U \cap V) \times N \to (U \cap V) \times N$ such that $\phi_V \circ \psi_{VU} = \phi_U$ on $(U \cap V) \times N$. Note that ψ_{VU} can be regarded as automorphsms of N parametrized by $U \cap V$. We can ψ_{VU} the *transition function* for the neighbourhoods U and V.

For compatibility it is necessary that the transition functions satisfy the conditions that, for $U \cap V \cap W \neq \emptyset$, we have that

(1) $\psi_{UU} = \mathbf{1}_{U \times N}$;
(2) $\psi_{UV} \circ \psi_{VU} = \mathbf{1}_{U \cap V \times N}$;
(3) $\psi_{UV} \circ \psi_{VW} \circ \psi_{WU} = \mathbf{1}_{U \cap V \cap W \times N}$.

1.2. **Vector bundle.** If instead of a generic manifold N, we let the fibre be a vector space, we have

Definition 5. A *vector bundle* is a fibre bundle (E, M, π, N) where the fibres are vector spaces.

Now, given two overlapping trivializations, the linear structure of N shows that the transition map must be linear along the fibre. So in particular, we can regard ψ_{VU} alternatively as a $GL(N)$ valued smooth map.

Example 2. The *tangent bundle* $\mathsf{T}M$ and *cotangent bundle* T^*M are specific examples of vector bundles.

1.3. $\mathcal{G}$-**structure.** In the case of the vector bundle, we see the appearance of the structure group $GL(N)$. This group arises as the natural symmetry group of the vector space N, the group of invertible mappings from N to itself that preserves the linear structure. Now, let U be an open subset of M and ϕ a local trivialisation. Suppose $A : U \to GL(N)$ is a smooth map (using that $GL(N)$ is a Lie group and hence is equipped with a smooth structure), then we can consider the diffeomorphism

$$\phi'(p,q) = \phi(p, A(p) \cdot q)$$

which defines another local trivialization of E over U that gives rise to the same linear structure of the fibres.

We can generalise this idea. Let $\mathcal{G}$ be a Lie group.

Definition 6. A *smooth fibre bundle with structure group* $\mathcal{G}$ is the quintuple $(E, M, \pi, N, \mathcal{G})$ where (E, M, π, N) is a smooth fibre bundle, $\mathcal{G}$ is a Lie group with acts smoothly and effectively on the fibre N with action $\mathcal{G} \ni g \mapsto A_g : N \to N$, and such that for any two overlapping local trivialisations (U, ϕ_U) and (V, ϕ_V), the transition function $\psi_{UV} : (U \cap V) \times N \to (U \cap V) \times N$ is given by

$$\psi_{UV}(p,q) = (p, A_{g(p)}q)$$

where $g : U \cap V \to \mathcal{G}$ is a smooth function.

Note that the requirement that the transition function is represented by $\mathcal{G}$-valued function g implies that A_g extends to a smooth and effective action on E that respects fibres.

Definition 7. A *smooth principal* $\mathcal{G}$ *bundle* is a smooth fibre bundle with structure group $\mathcal{G}$ where the action $g \mapsto A_g$ is free and transitive. In particular, this implies N is diffeomorphic to $\mathcal{G}$.

Frequently in applications the *gauge group* $\mathcal{G}$ represents the set of all smooth deformations of N that preserves a certain given structure.

Example 3. Going back to the example of a vector bundle (E, M, π, N), we can equip N with a non-degenerate, symmetric, scalar product $h : N \times N \to \mathbb{R}$. In this case, the given structure on the fibre is that they are scalar-product-spaces, and thus the natural symmetry group $\mathcal{G}$ is now the orthogonal group $O(N, h)$. Applied to the tangent bundle this gives rise to pseudo-Riemannian geometry.

1.4. **The momentum bundle.** This is a common construction especially in mathematical physics that it will be nice to incorporate here.

defn:mombun **Definition 8.** Given a fibred manifold (E, M, π), its associated *momentum bundle* over E is the pullback bundle (by π) cotangent bundle of M. We write it as $P = \pi^*(\mathsf{T}^*M)$.

1.5. **Bundle constructions.**

1.5.1. *Pullbacks and products.* Let (E_1, M, π_1, N_1) and (E_2, M, π_2, N_2) be two fibre bundles with the same base manifold. Consider the set

$$E = E_1 \times_M E_2 \stackrel{\text{def}}{=} \{\, (x_1, x_2) \in E_1 \times E_2 \mid \pi_1(x_1) = \pi_2(x_2) \,\}. \tag{2}$$

This set is a submanifold of $E_1 \times E_2$. This is due to the fact that the mapping $\pi_1 \times \pi_2 : E_1 \times E_2 \to M \times M$ is a submersion, and hence is transverse to any submanifold of $M \times M$. Now we now that the diagonal $M \subset M \times M$ is a regular submanifold, and so by a variant of the Preimage Theorem (using the transversality condition) the set $E = (\pi_1 \times \pi_2)^{-1}(M)$ is a submanifold.

Now, let $\rho_1 : E_1 \times E_2 \to E_1$ the projection onto the first factor, and analogously ρ_2. And let $\pi : E \to M$ be the projection $\pi = \pi_1 \circ \rho_1 = \pi_2 \circ \rho_2$. We claim that all three of π, $\rho_1|_E$, $\rho_2|_E$ are submersions. It is easy to see that it suffices to prove the claim for $\rho_1|_E$. Let $\rho_1|_E$ is a surjection follows from the fact that π_2 is a surjection. It remains to prove that $\mathsf{T}(\rho_1|_E) : \mathsf{T}E \to \mathsf{T}E_1$ is a surjection. But we simply verify that if $(x_1, x_2) \in E$, and $X_1 \in \mathsf{T}_{x_1}E_1$, by surjectivity of $\mathsf{T}\pi_2$ there exists $X_2 \in \mathsf{T}_{x_2}E_2$ such that $\mathsf{T}\pi_1(X_1) = \mathsf{T}\pi_2(X_2)$. But this implies that $X_1 + X_2 \in \mathsf{T}_{(x_1,x_2)}E \subset \mathsf{T}_{(x_1,x_2)}(E_1 \times E_2)$ showing surjectivity.

Remark 2. Notice that in the proof above that $\rho_1 : E \to E_1$ is a submersion depended only on π_2 being a submersion, and not on π_1. In particular, that fact stands even in the case π_1 is an arbitrary smooth map.

Remark 3. The construction of the set E and the maps ρ_1, ρ_2 is an example of the "categorical" pullback.

Definition 9. The *pullback bundle* of (E_2, M, π_2, N_2) onto E_1 via the smooth map $\pi_1 : E_1 \to M$ is the fibre bundle (E, E_1, ρ_1, N_2) constructed above. The fibres are diffeomorphic to N_2.

Definition 10. The *product bundle* of (E_1, M, π_1, N_1) and (E_2, M, π_2, N_2) is the fibre bundle $(E, M, \pi, N_1 \times N_2)$ constructed above.

Note that for the product bundle, we have that

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$$\mathsf{T}_{(x_1,x_2)}E = \left\{ (X_1, X_2) \in \mathsf{T}_{x_1}E_1 \times \mathsf{T}_{x_2}E_2 \;\middle|\; \mathsf{T}\pi_1(X_1) = \mathsf{T}\pi_2(X_2) \right\} \tag{3}$$

is a linear subspace of $\mathsf{T}_{(x_1,x_2)}(E_1 \times E_2)$.

1.5.2. *Dual bundle.* Given a vector bundle (E, M, π, N), there exists a dual bundle (E', M, π', N') where the fibres N' are the dual vector spaces of N. (Typically these are denoted N^* and E^*, but I do not want to confuse the dual projection map π' with the pullback map.) The actual construction of the dual bundle makes use of the Fibre Bundle Construction Theorem[2], taking the dual transition maps $\psi'_{ij}(p) = [(\psi_{ij}(p))^T]^{-1}$. (Note that the transpose operation maps elements of $GL(N)$ to elements of $GL(N')$; and the inversion is operator inversion in $GL(N')$. The cocycle condition for the bundle E now implies that the cocycle condition for ψ'_{ij} are satisfied.)

[2]This theorem states that given manifolds M, N, an open cover $\{U_i\}$ of M, and a collection of transition functions ψ_{ij} mapping $(U_i \cap U_j) \times N$ to itself, such that projection to the first factor commutes with ψ_{ij}, and such that the cocycle condition $\psi_{ij} \circ \psi_{jk} = \psi_{ik}$ holds, then there exists a total space E and a projection map π such that (E, M, π, N) is a fibre bundle with local trivialisations over the open sets U_i with the prescribed transition functions.

Remark 4. The cotangent bundle T^*M of a smooth manifold is the dual bundle of the tangent bundle $\mathsf{T}M$.

1.5.3. *Tensor products.* In the case of vector bundles, we can consider not only the Cartesian product, but also tensor products. The construction proceeds analogously as the product bundle case, except the fibre-wise Cartesian product is replaced by the fibre-wise tensor product. The easiest way to write it down, however, is through the Fibre Bundle Construction Theorem. Let (E_1, M, π_1, N_1) and (E_2, M, π_2, N_2) be (real) vector bundles. We can find a joint local trivialisation $\{(U_i, \phi_{1,i}, \phi_{2,i})\}$ for E_1 and E_2. Now take as the fibre manifold the tensor product $N_1 \otimes N_2$, and the transition maps $\psi_{1,ij} \otimes \psi_{2,ij}$ where recalling that $\psi_{1,ij}$ takes value in $GL(N_1)$ and $\psi_{2,ij}$ takes values in $GL(N_2)$. It is easy to see that the cocycle conditions are verified and so we can reconstruct a tensor bundle.

2. Derivatives

Recall now the elementary definition of derivatives sometimes encountered in first courses of calculus. Starting with $f : \mathbb{R} \to \mathbb{R}$, the derivative f' evaluated at a point x is the *slope* of the tangent line to f (supposing there exists a unique one) at the point x. Now the slope is defined as "rise over run", or "vertical displacement over horizontal displacement".

Now, we can identify the function f with its graph in $\mathbb{R}^2$. And we can treat the manifold $\mathbb{R}^2$ as a fibre bundle over $\mathbb{R}$. In particular, we forget about the identification of $\mathbb{R}^2$ with the Cartesian product $\mathbb{R} \times \mathbb{R}$. As a fibre bundle all we have access to is the projection map π.

From π itself we can not reconstruct what is "horizontal displacement". Let $g : \mathbb{R} \to \mathbb{R}$ be smooth. Let $(\mathbb{R}, \phi)$ be any trivialisation of $\mathbb{R}^2$ as a fibre bundle, we see that $(\mathbb{R}, \phi + g)$ is another trivialisation: here we made use of the structure of $\mathbb{R}$ as an additive Lie group. Relative to the two trivialisations, the "vertical part" of $\mathsf{T}\mathbb{R}^2$ is well-defined as the kernel of $\mathsf{T}\pi$. But the horizontal part, defined relative to the realisation of the trivialisation, are not the same. Thus for the function f which we now consider as a section of the fibre bundle $(\mathbb{R}^2, \mathbb{R}, \pi, \mathbb{R})$, we cannot associate a derivative.

In other words, in order to take derivatives what we need is to identify what it means for a tangent vector to be horizontal.

Definition 11. Let s be a section of E and X a section of $\mathsf{T}M$. A *horizontal lift* $\mathsf{H}(s, X)$ is a vector field along s such that $\mathsf{T}\pi(\mathsf{H}(s, X)) = X$.

Remark 5. Note that we do not assume that H is a linear map on T_pM for a fixed $p \in M$.

By definition we have that if s is a section of E, then $(\mathsf{T}s)(X) - \mathsf{H}(s, X) \in \ker \mathsf{T}\pi$, so we can define the vertical part of s in the direction of X as $(\mathsf{V}s)(X) = (\mathsf{T}s)(X) - \mathsf{H}(s, X)$. This is morally the "derivative" of s in the direction of X, relative to the chosen horizontal lift. The question then is: "how do we choose the horizontal lifts?"

2.1. **Vector bundles.** The above discussion works for general fibred manifolds, and in particular for fibre bundles. Note however that

Remark 6. The "derivative" takes value in $\mathsf{V}_x \subset \mathsf{T}_xE$ which is the tangent space of the fibre, and not the fibre itself.

This is slightly different from the usual definitions of covariant differentiation. Here we have to use the fact that for *vector* bundles, we can canonically identify $\mathsf{T}_q N$ with N.

Lemma 7. *Let N be a vector space and $w \in N$. The mapping $v \mapsto v + sw$ for $s \in \mathbb{R}$ is a one-parameter family of diffeomorphisms of N, and so its generator is a vector field on N. The mapping from w to the generator of the diffeomorphism, over a point v, is a linear isomorphism of N and $\mathsf{T}_v N$.*

Now, as remarked before, for a fibre bundle the vertical space $\mathsf{V} = \ker \mathsf{T}\pi \subset \mathsf{T}E$ is well defined.

cor:canidvert **Corollary 8.** *For a vector bundle, and any $x \in E$, we can (canonically) identify V_x with $E_{\pi(x)}$.*

2.2. **Product lift.** Given two fibre bundles (E_1, M, π_1, N_1) and (E_2, M, π_2, N_2), and two horizontal lifts $\mathsf{H}_1, \mathsf{H}_2$, what is the horizontal lift induced on the product bundle? This turns out to be pretty easy. In terms of the picture where $E = E_1 \times_M E_2 \subset E_1 \times E_2$, observe that the direct lift $(\mathsf{H}_1, \mathsf{H}_2)$ satisfies that for every X the projection from the two factors agree.

2.3. **Lie derivatives.** Let M be a manifold and X a vector field (section of $\mathsf{T}M$). The flow of X generates a one-parameter family of local diffeomorphisms (if M were compact or if X has suitable decay or bounds, then X generates a one-parameter family of diffeomorphisms) ${}^{(X)}_{s}\mathrm{Fl}$. We have the well-known definitions for Lie derivatives, where if σ is a section of $\mathsf{T}^k M$ (a contravariant tensor field) we define

$$\mathcal{L}_X \sigma \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}s} {}^{(X)}_{-s}\mathrm{Fl}_* \circ \sigma \circ {}^{(X)}_{s}\mathrm{Fl}\bigg|_{s=0}; \tag{4}$$

and for τ a section of $(\mathsf{T}^*)^k M$ (a covariant tensor field) we define

$$\mathcal{L}_X \tau \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}s} {}^{(X)}_{s}\mathrm{Fl}^* \circ \tau \bigg|_{s=0}. \tag{5}$$

We may ask, what are the corresponding horizontal spaces in $\mathsf{T}M$ and T^*M for these definitions?

Observe that ${}^{(X)}_{s}\mathrm{Fl}_*$ is a diffeomorphism from $E = \mathsf{T}M$ to itself that respects the projection map (in the sense that $\pi \circ {}^{(X)}_{s}\mathrm{Fl}_* = {}^{(X)}_{s}\mathrm{Fl} \circ \pi$). Since the flow forms a one parameter family we have that ${}^{(X)}_{s}\mathrm{Fl}_* \circ {}^{(X)}_{-s}\mathrm{Fl}_* = \mathbf{1}_{\mathsf{T}M}$, that is to say that the tangent flow also generates a one parameter family of diffeomorphisms. Its generator, which we denote by $\tilde{X}$, is a section of $\mathsf{TT}M$, or in other words a vector field over E. *This vector field $\tilde{X}$ is the requisite horizontal lift*, in that $\mathsf{H}(s, X) = \tilde{X} \circ s$.

Similarly, the operation ${}^{(X)}_{-s}\mathrm{Fl}^*$ is a diffeomorphism from $E = \mathsf{T}^*M$ to itself that respects the projection map (in the sense that $\pi \circ {}^{(X)}_{-s}\mathrm{Fl}^* = {}^{(X)}_{s}\mathrm{Fl} \circ \pi$); arguing as before we obtain a one parameter family of diffeomorphisms from which we can compute its generator, $\tilde{X}$, which is a section of $\mathsf{T}E = \mathsf{T}(\mathsf{T}^*M)$. *This vector field $\tilde{X}$ is the horizontal lift.*

Note that this point of view gives us a way to consider Lie differentiation of objects which naturally live in *sub-bundles* of the tangent or cotangent bundle (or their tensor products) of the manifold. For example, let $F \subset E = \mathsf{T}M$ be a fibred submanifold of E over M. We say that X is F-preserving if the lifted vector field $\tilde{X}|_F$

is tangent to F (this notion is only really meaningful if F has boundaries in E or if F is a positive-codimensional submanifold, for example). Then the Lie derivative $\mathcal{L}_X s$ of a section s of F can be interpreted as a section of $\mathsf{T}F$ over M without reference to the ambient manifold E.

2.4. **Connection.** When the fibred manifold (E, M, π) has no obvious relation between E and M, we cannot lift vector fields X on M to vector fields $\tilde{X}$ on E as in the previous section. This forces us to provide, externally, a notion of the horizontal lift. A particular notion of the lift is given by the concept of a *connection*.

Definition 12. Let (E, M, π) be a fibred manifold. A *horizontal distribution* is a smooth sub-bundle (over E) $\mathsf{H} \subset \mathsf{T}E$ such that for every $x \in E$, the fibre H_x is a linear sub-space of $\mathsf{T}_x E$ such that $\mathsf{T}_x E = \mathsf{H}_x \oplus \mathsf{V}_x$ where $\mathsf{V}_x \stackrel{\text{def}}{=} (\ker \mathsf{T}\pi) \cap \mathsf{T}_x E$.

With a horizontal distribution, the associated horizontal lift can be constructed fibre-wise: let $x \in E$ and $X \in \mathsf{T}_{\pi(x)} M$, the horizontal lift $\mathsf{H}(x, X)$ is the unique element of H_x satisfying $\mathsf{T}\pi(\mathsf{H}(x, X)) = X$.

Definition 13. Given a horizontal distribution H, its associated *connection* ∇ is the mapping

$$\nabla_X \sigma \stackrel{\text{def}}{=} (\mathsf{T}\sigma)(X) - \mathsf{H}(\sigma, X)$$

where X is a section of $\mathsf{T}M$ and σ is a section of E.

rmk:connection

Remark 9. Observe that

(1) $\nabla_X \sigma$ is a section of V treated as a bundle over M.
(2) $\nabla_X \sigma$ is tensorial in X, in that for f_1, f_2 smooth functions on M and X_1, X_2 sections of $\mathsf{T}M$ we have that $\nabla_{f_1 X_1 + f_2 X_2} \sigma = f_1 \nabla_{X_1} \sigma + f_2 \nabla_{X_2} \sigma$.
(3) rmk:connection:3 Suppose σ, τ are two sections of E, and $p \in M$ is such that $\sigma(p) = \tau(p)$, then $(\nabla_X)\sigma(p) - (\nabla_X \tau)(p) \in \mathsf{T}_{\sigma(p)} E$ is independent of the horizontal distribution H.

Let H and $\tilde{\mathsf{H}}$ be two horizontal distributions. By their definitions we have that $\mathsf{H}(x, X), \tilde{\mathsf{H}}(x, X) \in T_x E$ and so we can take their difference; by definition this difference lives in V_x. Noting further that this mapping is *linear* in X, we can regard the difference of H and $\tilde{\mathsf{H}}$, evaluated at a point $x \in E$, as a linear map from $\mathsf{T}_{\pi(x)} M$ to V_x. So considering the total space

$$\cup_{x \in E} \mathsf{V}_x \otimes \mathsf{T}^*_{\pi(x)} M = \mathsf{V} \otimes P$$

given by the tensor product of the vertical bundle and the momentum bundle (see Definition 8), with the base manifold E and the natural projection, the difference of two connections is naturally a section of this bundle.

Definition 14. The *relative connection coefficients* between the horizontal distributions H and $\tilde{H}$ is the section of $\mathsf{V} \otimes P$ over E described above.

2.4.1. *Bundles and local representations.* Now we restrict our focus to the case of a fibre bundle (E, M, π, N). Let (U, ϕ) be a local trivialisation of the bundle, a choice of horizontal distribution arises from the Cartesion product structure $U \times N$: there exists the canonical splitting $\mathsf{T}_{(p,q)}(U \times N) = \mathsf{T}_p U \oplus \mathsf{T}_q N$ so *associated to the trivialisation* (U, ϕ), we can define a horizontal distribution by

$$\mathsf{H}_x = \phi_*(\mathsf{T}_{\pi(x)} U).$$

Definition 15. Given a local trivialisation (U,ϕ) of a fibre bundle, the *(local) coordinate connection* associated to (U,ϕ) is denoted by ∂, and is the connection given by the horizontal distribution $\phi_*(\mathsf{T}U)$.

This in general does not extend to a global definition: for two different overlapping local trivialisations, the horizontal distributions defined by the two trivialisation maps on the overlap usually do not agree.

Remark 10. Let σ be a section of E. Over U using the trivialisation $\sigma \circ \phi$ can be regarded as a mapping $U \to N$. Then the coordinate connection is simply

$$\partial(\sigma \circ \phi) = \mathsf{T}(\sigma \circ \phi).$$

Definition 16. The *(local) connection coefficients* of a horizontal distribution H relative to the trivialisation (U,ϕ) is by definition the relative connection coefficients between H and $\phi_*(\mathsf{T}U)$.

2.4.2. *Linear connection.* In the case where (E,M,π,N) is a vector bundle, so that the fibres have a linear structure, We can demand that a connection observe (some portion of) the linear structure. Here and throughout we identify $\nabla_X\sigma$ for a section X of $\mathsf{T}M$ and a section σ of E with a section of E via Corollary 8.

Definition 17. A connection ∇ is said to be *Leibniz* if it follows the Leibniz rule of scalar multiplication; that is to say, if σ is a section of E and f a smooth function on M, for the section $f\sigma$ we have that

$$\nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma.$$

Observe now that for a vector bundle E, there is a privileged global section $\vec{\mathbf{0}}$, using that in a vector space the $\vec{0}$ vector is privileged. It is easy to see that the zero section must satisfy $\nabla_X\vec{\mathbf{0}} = \vec{\mathbf{0}}$ for any vector field X, whenever ∇ is Leibniz. In other words, H is exactly $\mathsf{T}\vec{\mathbf{0}}(\mathsf{T}M)$. Therefore we have

Lemma 11. *Let ∇ and $\tilde{\nabla}$ be two Leibniz connections on the vector bundle (E,M,π,N). For any section σ and any p such that $\sigma(p) = \vec{0}$, we must have $\nabla_X\sigma(p) = \tilde{\nabla}_X\sigma(p)$.*

Definition 18. A connection ∇ is said to be *linear* if for every pair of sections σ,τ of E, we have

$$\nabla_X(\sigma+\tau) = \nabla_X\sigma + \nabla_X\tau.$$

The following proposition is simple to prove.

Proposition 12. *The local coordinate connection of a vector bundle is linear.*

Now let ∇ and $\tilde{\nabla}$ be two linear connections. We have that their relative connection coefficients are linear in the fibre N. Therefore we have

Proposition 13. *The relative connection coefficients between two linear connections on (E,M,π,N) is a section of $E\otimes E^*\otimes \mathsf{T}^*M$ over M. In particular, the local connection coefficients of a linear connection relative to a trivialisation (U,ϕ) can be identified with a $GL(N)$ valued one form over M.*

We can also take tensor products of linear connections. One way to describe the resulting connection is through the connection coefficients relative to a local coordinate. Let (E_i,M,π_i,N_i) with $i = 1,2$ two vector bundles, and $E = E_1\otimes E_2$. Suppose we have two linear connections ∇_1,∇_2 on the two bundles such that their local connection coefficients are represented by the $GL(N_i)$ valued one forms ω_i. Then the local connection coefficients of the tensor-product connection ω can be expressed as

$$\omega(X) = \omega_1(X) \otimes \mathbf{1}_2 + \mathbf{1}_1 \otimes \omega_2(X) \tag{6}$$

for every tangent vector X.

2.4.3. *$\mathcal{G}$-invariant connections.* Now moving to the case of fibre bundles with structure group $\mathcal{G}$, we can add the requirement that the horizontal distributions are invariant under the global group action $A_g : E \to E$ (more precisely, its pushforward map). In the case of principal $\mathcal{G}$-bundles, this also means that a specification of H at one point along each fibre is enough to specify the connection completely.

Now, in the case of principal $\mathcal{G}$-bundles, we can use the $\mathcal{G}$ action to identify the tangent spaces T_xE and $T_{A_g(x)}E$ using the diffeomorphism $\mathsf{T}A_g : \mathsf{T}E \to \mathsf{T}E$. On the other hand, the vertical fibre can be identified with $\mathsf{T}N$ which is diffeomorphic to $\mathsf{T}\mathcal{G}$. Thus, *so long as we choose a point on each fibre E_p to be "the identity", we can identify every V_x with the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$.*

Therefore, we have the familiar fact that for a fixed choice of a local section σ representing "the identity", a $\mathcal{G}$-invariant connection can be represented as a $\mathfrak{g}$ valued one-form over M. Of course, two different choices of σ leads to different local representations; the change of variable formula depends on whether A_g is a left or right action, and can be find in must texts on Yang-Mills theory.

2.5. **Covariant Lie derivative.** Now we arrive at something less often seen in introductory texts, but is entirely natural. Let (E, M, π) be an arbitrary fibred manifold, and let H be an arbitrary horizontal distribution. Since this horizontal distribution is *fibre-wise defined*, we can then define the *horizontal lift* relative to this distribution of any section of $\mathsf{T}M$. More precisely, we have that

eqn:def:cnxhorlift

$$\tilde{X} = H(\bullet, X) \tag{7}$$

is a section of $\mathsf{T}E$ which projects to X along M. This allows us to define the *Lie covariant derivative* of sections of $\mathsf{T}E$ or T^*E relative to a vector field on the base M.

What's more useful, however, is the case of the vector bundle. Let (E, M, π, N) be a vector bundle, denote by $\mathsf{T}^{p,q}M = (\otimes^p \mathsf{T}M) \otimes (\otimes^q \mathsf{T}M)$ some tensor power of the tangent and cotangent bundle over M. Observe that the usual Lie derivative is linear in sections:

$$\mathcal{L}_X(\sigma + \tau) = \mathcal{L}_X\sigma + \mathcal{L}_X\tau$$

for σ, τ sections of $\mathsf{T}^{p,q}M$. So in particular in a local trivialisation we can write

$$\mathcal{L}_X\sigma = \partial_X\sigma + \omega(X) \cdot \sigma$$

where $\omega(X)$ is a $GL(\mathsf{T}^{p,q}M)$ valued function on M. Thus we can take the tensor product between this and the covariant derivative relative to X (where the local connection coefficients contracted with X gives a $GL(N)$ valued function on M). This gives the coefficients for our "covariant Lie derivative" $\mathcal{L}^{\nabla}_X$.

Alternatively, we can under the covariant Lie differentiation as follows. Lifting the vector field X to $\tilde{X}$ allows us to extend ${}^{(X)}_{s}\mathrm{Fl}$ to a one parameter family of diffeomorphisms of E that commutes with the projection π. The fact that ∇ is a linear connection implies that the action of ${}^{(X)}_{s}\mathrm{Fl}$ on the fibres are linear: this can be seen most easily within a local trivialisation. On the other hand, the pushforward

Perhaps extract this construction and rewrite it?

and pullback maps, as discussed before, of ${}^{(X)}_{s}\mathrm{Fl}$ lifts to diffeomorphisms of TM and T^*M that commutes with projections, and are linear in the fibres. Therefore we can take the tensor product of the diffeomorphisms to get a one-parameter family of diffeomorphisms induced by ${}^{(X)}_{s}\mathrm{Fl}$ through the linear connection ∇ on $\mathsf{T}^{p,q}M \otimes E$. The *generator* of this flow is the induced horizontal lift by, on the one hand, Lie transport on the base manifold and its tangent/cotangent bundles and on the other hand, parallel transport by ∇ in E. And this is precisely the horizontal lift that is associated to the covariant Lie derivative.

3. Curvature

The classical formulae for Riemann curvature or curvature in principal G-bundles (Yang-Mills theory) is well known. Here we give a description of how to attach the curvature to an arbitrary connection in a fibre bundle.

Let (E, M, π) be a fibred manifold, and let H be a horizontal distribution. Now consider X, Y vector fields along M. The horizontal distribution allows us to lift X, Y to vector fields $\mathsf{H}(X)$ and $\mathsf{H}(Y)$ on E.

Definition 19. The *curvature* of the horizontal distribution H is the assignment, for X, Y sections of $\mathsf{T}M$, the section of $\mathsf{T}E$

$$\mathcal{R}(X,Y) \stackrel{\text{def}}{=} [\mathsf{H}(X), \mathsf{H}(Y)] - \mathsf{H}([X,Y]).$$

Proposition 14. *The curvature $\mathcal{R}$ satisfies, for X, Y sections of $\mathsf{T}M$ and f, g scalar functions,*

(1) $\mathsf{T}\pi(\mathcal{R}(X,Y)) = 0$;
(2) $\mathcal{R}(X,Y) = -\mathcal{R}(Y,X)$;
(3) $\mathcal{R}(fX, gY) = fg\mathcal{R}(X,Y)$.

Proof. (1) We use that π is a smooth map and that push forwards commute with Lie brackets: given any smooth map ϕ and vector fields v, w we have $\phi_*[v,w] = [\phi_* v, \phi_* w]$.

(2) We use that the Lie brackets are anti-symmetric, and that $\mathsf{H}_x \mathsf{T}_{\pi(x)} M \to \mathsf{T}_x E$ is linear.

(3) By linearity of H_x, we have that $\mathsf{H}(fX) = f\mathsf{H}(X)$. Thus

$$\begin{aligned}(8)\quad [\mathsf{H}(fX), \mathsf{H}(gY)] &= (\pi^* f)(\pi^* g)[\mathsf{H}(X), \mathsf{H}(Y)] \\ &\quad + (\mathsf{H}(X) \cdot \mathrm{d}(\pi^* g))(\pi^* f)\mathsf{H}(Y) - (\mathsf{H}(Y) \cdot \mathrm{d}(\pi^* f))(\pi^* g)\mathsf{H}(X).\end{aligned}$$

Note that $\mathsf{H}(X) \cdot \mathrm{d}(f \circ \pi) = X \cdot \mathrm{d}f$. On the other hand we have that

$$\mathsf{H}([fX, gY]) = \mathsf{H}(fg[X,Y] + X(g)fY - Y(f)gX)$$

so we have exactly the claim.

□

In other words, we can describe $\mathcal{R}$ as a section of $\Lambda^2 E \otimes \mathsf{V}$ which annihilates V. Note that in the case where H is a linear connection in a vector bundle, the dependence of $\mathcal{R}$ on E_p is linear, and identifying V_x with $E_{\pi(x)}$ we can take $\mathcal{R}$ to be a $GL(N)$ valued two-form over M. In the case where H is a $\mathcal{G}$-invariant connection, fixing a local section σ of the principal $\mathcal{G}$-bundle, we are left with a representation of $\mathcal{R}$ as a $\mathfrak{g}$ valued two-form over M.

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