MTH 847: PDE I *(Fall 2017)* ***Exam 2, 2017.12.8***

Name:

Standard exam rules apply:

- You are not allowed to give or receive help from other students.
- All electronic devices must be turned **off** for the duration of the test and stowed. This includes phones, pagers, laptops, tablets, e-readers, and calculators.
- The only things allowed on your desk are:
  - Your writing implements, including also corrector fluids or erasers or similar.
  - A water bottle or other drink.
  - This booklet.
- This exam lasts from 11:30 – 12:20. Students may not leave the room until after 12:00; students may not (re)enter the room after 12:00.

INSTRUCTOR USE ONLY:

Q#	pts	MAX
1A		4
1B		4
2A		2
2B		2
3A		4
3B		4
4		4
TOTAL		24

Q1. Let $\phi \in C^2(\mathbb{R} \times \mathbb{R}^3)$ be a solution to the linear wave equation $\Box\phi = 0$. Let $\lambda \in \mathbb{R}_+$ and denote by E the set

$$E := \{(t, x) \in \mathbb{R} \times \mathbb{R}^3 \mid |t| \leq \lambda, \ \ |x| \geq 1\}.$$

A. (4pts) Suppose $\lambda < 1$. Show that there exists a solution ϕ to the wave equation, that is not identically zero, but vanishes on the set E.

B. (4pts) Suppose $\lambda > 1$. Show that any solution ϕ that vanishes on the set E must be identically zero.
(Hint: strong Huygens' principle [alternatively the Kirchhoff formula] can be useful.)

Solutions.

A. Let $\epsilon < 1-\lambda$. Take any $g \in C_c^\infty(B(0,\epsilon))$ and solve the wave equation with initial data $\phi(0,x) = g(x)$ and $\partial_t\phi(0,x) = 0$. By the finite speed of propagation property we have that $\phi \equiv 0$ whenever $|x| \geq |t| + \epsilon$. Our initial choice of ϵ then guarantees that $E \subset \{\phi = 0\}$. If g is chosen to be nontrivial, the solution is also nontrivial.

B. Consider the points (t,x) with $t = 1$ and $|x| < 1$. Denote by $g(x) = \phi(-1,x)$ and $h(x) = \partial_t\phi(-1,x)$. By the Kirchoff formula we have that

$$\phi(1,x) = \frac{1}{4\pi}\int_{\mathbb{S}^2} g(x+2\omega) + 2\partial_\nu g(x+2\omega) + 2h(x+2\omega)\,\mathrm{d}\omega.$$

Notice that when $|x| < 1$, we have that $|x+2\omega| > 1$ by triangle inequality, and hence $(-1, x+2\omega) \in E$. This means that the integrand vanishes by assumption. Therefore $\phi(1,x) \equiv 0$ for all x. Similarly we conclude that $\partial_t\phi(1,x) \equiv 0$ for all x, and hence ϕ is a solution to the linear wave equation such that $\phi(1,x) = \partial_t\phi(1,x) \equiv 0$ and by uniqueness of solutions we have that $\phi \equiv 0$.

Q2. Suppose $\phi \in C^2(\mathbb{R} \times \mathbb{R})$ solves

$$
\begin{gathered}
-\partial^2_{tt}\phi + \partial^2_{xx}\phi = 0 \\
\phi(0,x) = 0 \\
\partial_t\phi(0,x) = h(x)
\end{gathered}
$$

where the data $h \in C^2(\mathbb{R})$ is such that $h(x) = 0$ for all $|x| \geq 1$; and $h(x) > 0$ for all $|x| < 1$.

A. (2pts) Find all space-time points (t,x) where $\phi(t,x) = 0$.
B. (2pts) Find all space-time points (t,x) where $\partial_t\phi(t,x) = 0$.

Solutions. The answer to both parts follow from applying D'Alembert's formula

$$\phi(t,x) = \frac{1}{2}[g(x+t) + g(x-t) + \int_{x-t}^{x+t} h(y)\, \mathrm{d}y].$$

A. In this part $g(x) = 0$. So when $|x| \geq |t| + 1$ we have that $h(y)$ vanishes in the integrand, and hence $\phi(t,x) = 0$ there. On the other hand, when $|x| < |t| + 1$, $\phi(t,x)$ is equal to the integral of a non-trivial, non-negative continuous function, and hence is positive.

B. Setting $\psi = \partial_t \phi$, we have that ψ solves the wave equation with data $\psi(0,x) = h(x)$ and $\partial_t \psi(0,x) = 0$. So applying D'Alembert's formula we have that

$$\partial_t \phi(t,x) = \frac{1}{2}[h(x+t) + h(x-t)].$$

Hence $\partial_t \phi(t,x) > 0$ whenever $(t,x) \in \{|x+t| < 1\} \cup \{|x-t| < 1\}$, and vanishes exactly on its complement.

Q3. Consider the Cauchy problem

$$\partial_t \phi + x \partial_x \phi = 0$$
$$\phi|_\Sigma = f$$

where $(t, x) \in \mathbb{R} \times \mathbb{R}$, and $\Sigma = \{t = |x|\}$.

A. (4pts) Give an example of a function $f \in C^1(\mathbb{R} \times \mathbb{R})$ such that there does not exist a $C^1(\mathbb{R} \times \mathbb{R})$ solution to the Cauchy problem. Justify your example.

B. (4pts) Give an example of a function $f \in C^1(\mathbb{R} \times \mathbb{R})$ such that there exists infinitely many $C^1(\mathbb{R} \times \mathbb{R})$ solutions to the Cauchy problem. Justify your example.

Solutions.

A. The corresponding integral curves are $x = Ce^t$ for $C \in \mathbb{R}$. Hence there are multiple ways to answer this question.

   1. We can use the fact that Σ is not smooth: if we let $f(t,x) = t$, then if ϕ is any C^1 function agreeing with f along Σ, we must have $\partial_t\phi(0,0)+\partial_x\phi(0,0) = \partial_t\phi(0,0)-\partial_x\phi(0,0) = 1$. This means that the equation $\partial_t\phi + x\partial_x\phi$ cannot hold at the origin.
   2. We can use the fact that the integral curves intersect Σ more than once: Let $C = e^{-2}$ for example, then the system $x = e^{t-2}$ and $|x| = t$ has two distinct solutions with different values of t. So setting $f(t,x) = t$ again means that we cannot have a solution, since the PDE implies ϕ must be constant along the integral curves.
   3. We can use the fact that the integral curve becomes tangent to Σ when $C = \pm e^{-1}$. Here $(t,x) = (1,\pm 1)$. Setting again $f(t,x) = t$ would require that at $(t,x) = (1,\pm 1)$ that $\partial_t\phi(1,\pm 1) + x\partial_x\phi(1,\pm 1) = 1$, ruling out the existence of a solution.

B. Let $\psi \in C_c^\infty(\mathbb{R})$. Then we can check that setting $\phi(t,x) = \psi(t - \ln|x|)$ we have

$$\partial_t\phi + x\partial_x\phi = \psi'(t-\ln|x|)\cdot(1 - x\cdot\frac{1}{x}) = 0.$$

Furthermore, since ψ has compact support we have that ϕ is also smooth (especially near $|x| = 0$). Next observe that if $t = |x|$, then $t - \ln|x| = t - \ln t > 0$. So in particular if ψ and $\tilde{\psi}$ agree on the positive real axis, then the corresponding ϕ and $\tilde{\phi}$ agree on the set Σ. So letting $f = \phi = \psi(t - \ln|x|)$ for any ψ described as above, we see that there exists infinitely many solutions to the Cauchy problem.

Q4. (4pts) Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with C^1 boundary. Consider the system

$$\Box\phi = 0$$

on $(0,T)\times\Omega$ with the nonlinear initial-boundary conditions

$$\phi(0,x) = \partial_t\phi(0,x) = 0, \quad x \in \Omega;$$
$$(\partial_t\phi)^3 + e^{\phi}\partial_n\phi = 0, \quad \text{along } [0,T]\times\partial\Omega.$$

Here $\partial_n\phi$ denotes the outward normal derivative on $\partial\Omega$. Prove that if $\phi \in C^2([0,T]\times\overline{\Omega};\mathbb{R})$ solves the system above, then $\phi \equiv 0$ on $[0,T]\times\overline{\Omega}$.

Solutions.

The energy method gives, for every $\tau \in (0, T]$,

$$0 = \int_{\{\tau\}\times\Omega} \nu \cdot {}^{(\partial_t)}J \, \mathrm{d}S + \int_{\{0\}\times\Omega} \nu \cdot {}^{(\partial_t)}J \, \mathrm{d}S + \int_{[0,\tau]\times\partial\Omega} \nu \cdot {}^{(\partial_t)}J \, \mathrm{d}S = E_\tau + E_0 + F.$$

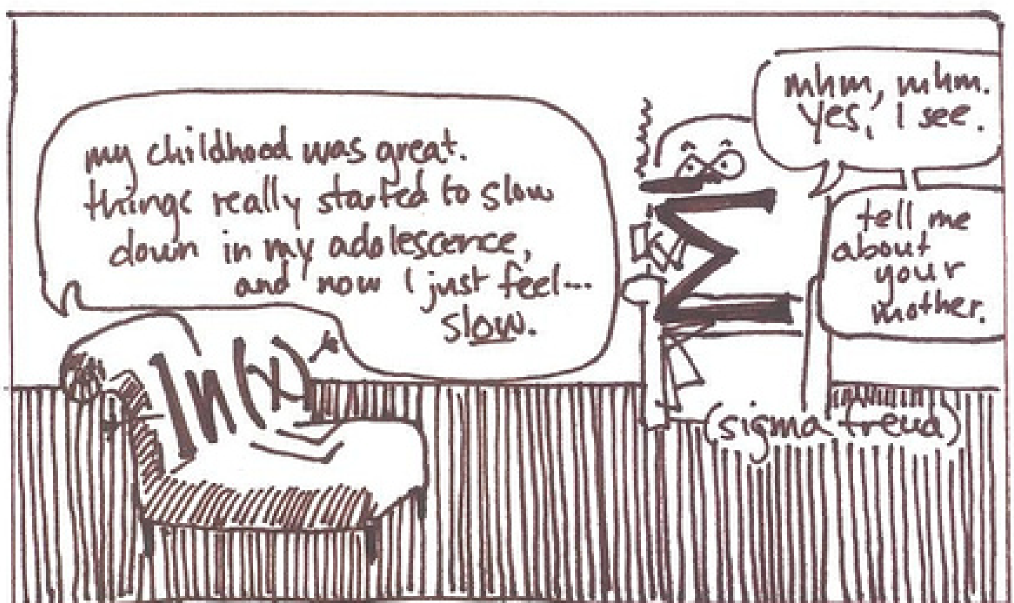
Our choice of initial data means that E_0, the integral over $\{0\} \times \Omega$, vanishes. We know that along $\{\tau\} \times \Omega$, the outward normal $\nu = \partial_t$ and so $E_\tau \leq 0$.

We can compute on $[0, \tau] \times \partial\Omega$

$${}^{(\partial_t)}J \cdot \nu = \partial_n \cdot m \cdot Q \cdot \partial_t = \partial_n\phi\partial_t\phi \leq 0$$

since the boundary assumption implies that $\partial_t\phi$ and $\partial_n\phi$ have opposite signs.

Together this implies that $E_\tau = F = 0$ for any τ. This means that $\nabla\phi \equiv 0$ on $[0, T] \times \Omega$ and hence $\phi \equiv 0$.



FUNCTIONAL ANALYSIS

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