MTH 847: PDE I *(Fall 2017)* ***Exam 1, 2017.10.18***

Name:

Standard exam rules apply:

- You are not allowed to give or receive help from other students.
- All electronic devices must be turned **off** for the duration of the test and stowed. This includes phones, pagers, laptops, tablets, e-readers, and calculators.
- The only things allowed on your desk are:
  - Your writing implements, including also corrector fluids or erasers or similar.
  - A water bottle or other drink.
  - This booklet.
- This exam lasts from 11:30 – 12:20. Students may not leave the room until after 12:00; students may not (re)enter the room after 12:00.

INSTRUCTOR USE ONLY:

Q#	pts	MAX
1		5
2A		4
2B		4
3A		4
3B		4
4		4
TOTAL		25

Q1. Let Ω be a bounded, open domain in $\mathbb{R}^d$. Suppose $u \in C^2(\overline{\Omega})$ solves the equation

$$-\triangle u + |\nabla u|^2 = 0 \qquad \text{on} \Omega$$

with boundary condition

$$u \equiv 0 \qquad \text{on} \partial\Omega.$$

- (5pts) Prove that if $u \geq 0$, then $u \equiv 0$. (If you use a maximum principle, make sure to include the precise statement of your maximum principle and give a brief [no more than 4 sentences] justification of why the maximum principle holds.)
- (Extra credit 2pts) Show that, even without the assumption $u \geq 0$, we can conclude that $u \equiv 0$.

Solutions. One option is to appeal to the maximum principle for subharmonic functions discussed on question 8 of Homework 2. The principle states that

> If $u \in C^2(\overline{\Omega})$ satisfies $-\triangle u \leq 0$, then $\max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.

This maximum principle can be proven by using the mean value property. Defining

$$\phi(r) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u(y) \, \mathrm{d}S(y)$$

we can compute to show $\phi(0) = u(x)$ and

$$\phi'(r) = \frac{1}{|\partial B(x,r)|} \int_{B(x,r)} \triangle u(y) \, \mathrm{d}y \geq 0.$$

This implies via a continuity argument that if u attains its maximum in the interior of Ω, then u must be constant on the corresponding connected component.

By the maximum principle we have that $u \leq 0$, but by assumption we have $u \geq 0$, hence $u \equiv 0$.

Extra credit: for the extra credit, notice that defining $v = e^{-u}$ we have that $\nabla v = -e^{-u}\nabla u$ and so $\triangle v = e^{-u}(-\triangle u + |\nabla u|^2) = 0$. Apply both the maximum and minimum principle to v we conclude that v is constant. And hence so must be u.

Q2. Let Ω be a bounded, open domain in $\mathbb{R}^d$ with C^1 boundary. Let $g \in C^0(\partial\Omega)$. Define

$$\mathcal{A} = \{u \in C^2(\overline{\Omega}) : u = g \text{ on } \partial\Omega\}.$$

A. (4pts) Let $P(z)$ be a polynomial function of z. Define

$$I_P[u] = \int_\Omega \frac{1}{2}|\nabla u|^2 + P \circ u \, \mathrm{d}x.$$

Prove that: if $w \in \mathcal{A}$ is such that

$$I_P[w] = \inf_{v \in \mathcal{A}} I_P[v],$$

then w solves

$$-\triangle w + P'(w) = 0.$$

B. (4pts) Prove that the converse to part A is not necessarily true. More precisely, given $P(z) = z^3$ and $g \equiv 0$, show that there exists a solution $w \in \mathcal{A}$ of

$$-\triangle w + P'(w) = 0$$

such that w does not minimize I_P.

(Hint for both parts: consider $I_P[w + \lambda v]$ for $\lambda \in \mathbb{R}$ and v a fixed function.)

Solutions.

A. Let $v \in C_c^\infty(\Omega)$ be fixed but arbitrary, and consider $I_P[w+tv]$ for $t \in \mathbb{R}$. Notice that $I_P[w+tv]$ is a polynomial in t, and hence is differentiable. By assumption $t=0$ is a global minimum to this polynomial, and so by elementary calculus $\frac{d}{dt} I_P[w+tv]\Big|_{t=0} = 0$.

   A direct computation shows that this implies

$$\int_\Omega \nabla w \cdot \nabla v + P'(w) v \, \mathrm{d}x = 0.$$

   Integration by parts we get

$$\int_\Omega v[-\triangle w + P'(w)] \, \mathrm{d}x = 0.$$

   Since this holds for all $v \in C_c^\infty(\Omega)$, we conclude that $-\triangle w + P'(w)$ must vanish.

B. Observe that $w \equiv 0$ is in $\mathcal{A}$, and solves the PDE. Now let v be any non-zero, non-negative $C_c^\infty(\Omega)$ function. Then

$$I_P[w+tv] = \int_\Omega \frac{1}{2} |\nabla v|^2 t^2 + t^3 v^3 \, \mathrm{d}x.$$

   This is a polynomial of the form $At^3 + Bt^2$ with $A > 0$, so there exists some $T < 0$ such that $I_P[w+Tv] < 0 = I_P[0]$.

Q3. Consider the initial value problem

$$\partial_t u(t,x) - \triangle u(t,x) + f(t)u(t,x) = 0$$

on $(0,\infty)\times\mathbb{R}^d$, with initial value $u(0,x) = g(x)$. The functions f and g are both assumed to be continuous, and g is further assumed to be bounded.

A. (4pts) Write down an explicit formula for the solution u in terms of f and g.
B. (4pts) Suppose there exists $\lambda > 0$ such that $f(t) \geq \lambda$ for all $t > 0$. Prove that

$$\sup_{x\in\mathbb{R}^d} |u(t,x)| \leq e^{-\lambda t} \sup_{x\in\mathbb{R}^d} |g(x)|.$$

Solutions. Let $F(t) = \int_0^t f(s)\, \mathrm{d}s$; this quantity is differentiable and well-defined as f is continuous. Consider the function $w(t,x) = e^{F(t)}u(t,x)$. We see that

$$\partial_t w - \triangle w = e^{F(t)}(\partial_t u + fu - \triangle u)$$

and $w(0,x) = u(0,x)$, so if w solves the heat equation with initial data $g(x)$, then $u = e^{-F}w$ solves the desired PDE.

Now let $\Phi(t,x)$ denote the fundamental solution to the heat equation, we can write

$$w(t,x) = \int_{\mathbb{R}^d} \Phi(t, x-y) g(y)\, \mathrm{d}y$$

and so

$$u(t,x) = e^{-\int_0^t f(s)\, \mathrm{d}s} \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}} e^{-|x-y|^2/4t} g(y)\, \mathrm{d}y.$$

The assumption that $f(t) \geq \lambda$ implies $F(t) \geq \lambda t$. The weak maximum principle applied to w (since g is bounded) implies

$$\sup_{x\in\mathbb{R}^d} |w(t,x)| \leq \sup_{x\in\mathbb{R}^d} |g(x)|.$$

The desired inequality follows since

$$\sup_{x\in\mathbb{R}^d} |u(t,x)| = e^{-F(t)} \sup_{x\in\mathbb{R}^d} |w(t,x)| \leq e^{-\lambda t} \sup_{x\in\mathbb{R}^d} |g(x)|.$$

Q4. Let $u_1 \leq u_2 \leq u_3 \leq \cdots$ be a sequence of *increasing* harmonic functions defined on $B(0,1) \subset \mathbb{R}^d$.

- (4pts) Prove that if there exists $x \in B(0,1)$ such that $\lim_{i\to\infty} u_i(x) = +\infty$, then *for every* $y \in B(0,1)$, the limit $\lim_{i\to\infty} u_i(y) = +\infty$.
  (Hint: Harnack's inequality.)
- (Extra credit 3pts) Suppose there exists some $M > 0$ and a point x such that $u_i(x) \leq M$ for every i. Prove that the pointwise limit $u(y) := \lim_{i\to\infty} u_i(y)$ of the sequence (u_i) exists and is harmonic on $B(0,1)$.

Solutions.

Harnack's inequality states, in this context, that

> If E is a bounded connected subset of $B(0,1)$ such that $\overline{E} \subset B(0,1)$, then there exists a constant C such that for every non-negative harmonic function v, $\sup_E v \leq C \inf_E v$.

Without loss of generality we can assume $u_1 \equiv 0$; otherwise consider instead the sequence $\tilde{u}_i = u_i - u_1$ of increasing non-negative harmonic functions.

Let y be arbitrary. Let R be such that $\max(|x|,|y|) < R < 1$, then we can set $E = B(0,R)$ so that $x, y \in E$. Harnack's inequality implies

$$u_i(x) \leq \sup_E u_i \leq C \inf_E u_i \leq C u_i(y)$$

where C depends only on R and not on i. Then as $u_i(x)$ diverges to infinity as $i \to \infty$, so must $u_i(y)$.

For the extra credit: note that the statement implies its converse that if for some x the values $u_i(x)$ is bounded for all i, then $u_i(y)$ is bounded (with a different bound) for all y. As an increasing bounded sequence must converge, we have established pointwise convergence of the sequence (u_i).

To prove that u is harmonic, it suffices to prove that it is continuous and satisfies the mean value property. Observe that as a consequence of Harnack's inequality, for every $R \in (0,1)$, the functions u and u_i are all uniformly bounded on $B(0,R)$. Now let $y \in B(0,R)$ be arbitrary, and let $z \in B(0,R)$ be a nearby point. We have by the mean value property

$$u_i(y) - u_i(z) = \frac{1}{|B(0,r)|}\left[\int_{B(y,r)} u_i(\xi)\,\mathrm{d}\xi - \int_{B(z,r)} u_i(\xi)\,\mathrm{d}\xi\right].$$

This implies

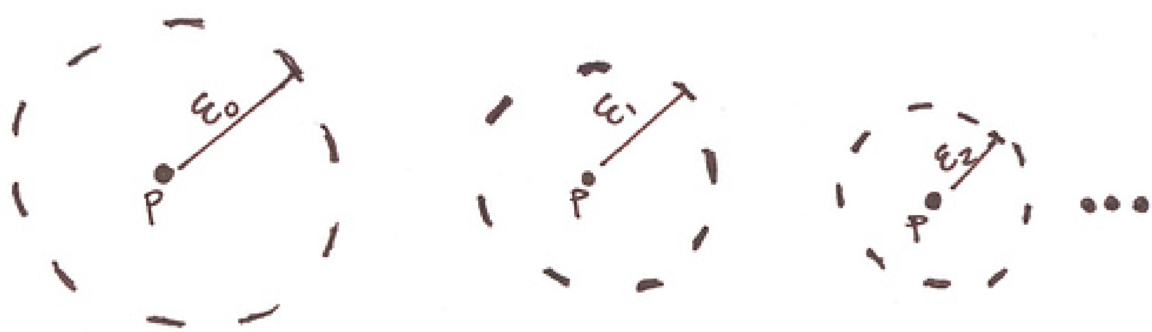
$$|u_i(y) - u_i(z)| \leq \frac{|B(y,r)\Delta B(z,r)|}{|B(0,r)|} \max_{B(0,R)} |u|$$

provided r is sufficiently small that $B(y,r) \cup B(z,r) \subset B(0,R)$. ($\Delta$ denotes the symmetric set difference.) Notice that the factor

$$\frac{|B(y,r)\Delta B(z,r)|}{|B(0,r)|} = O(|y-z|)$$

in this case.

This implies that given y, for every ϵ, there exists some $\delta > 0$ such that if $|y-z| < \delta$ then for every $i \in \mathbb{N}$ we have the uniform estimate $|u_i(y) - u_i(z)| < \epsilon/3$. (In other words, the sequence u_i is equicontinuous on $B(0,R)$.) Therefore for any $R \in (0,1)$, the sequence u_i is uniformly equicontinuous on $\overline{B(0,R)}$, and thus by Arzelà-Ascoli theorem, the pointwise convergence of u_i is in fact uniform convergence, and therefore u is continuous and satisfies the mean value property, and hence is harmonic.



There goes the neighborhood.