Energy method for wave equations
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Abstract We give an elementary discussion of the energy method (and particularly the vector field method) in the context of linear wave equations on $\mathbb{R} \times \mathbb{R}^d$.

The goal of this short note is to describe the modern approach to the energy method for, generally speaking, nonlinear and variable-coefficient wave equations, by exposing the theory in the context of the linear wave equations on $\mathbb{R} \times \mathbb{R}^d$.

Our main object of study will be solutions $\phi \in C^2(\mathbb{R} \times \mathbb{R}^d; \mathbb{R})$ to the equation

$$\Box \phi = 0$$

$$\phi(0, x) = g(x)$$
$$\partial_t \phi(0, x) = h(x)$$

where the couple of real-valued functions $(g, h)$ on $\mathbb{R}^d$ is referred to as the initial conditions. The $\Box$ operator, as a reminder, is $-\partial^2_{tt} + \Delta$ where $\Delta$ is the Laplacian on $\mathbb{R}^d$.

The energy method is an $L^2$-based method; in other words, we study integral quantities related to the square norm of the solution. In particular, we will be particularly concerned with integrals over certain hypersurfaces in $\mathbb{R} \times \mathbb{R}^d$; for this their geometries play important roles. Most modern discussions of the energy method in wave equations in fact proceed in the settings Lorentzian geometry, which is the natural setting for these discussions. However, as a first introduction, we will present explicit formulae in these notes and forego the high-powered geometric techniques in favor of direct computations. This hopefully makes the gist of the arguments more approachable.

1. The energy momentum tensor

For convenience, we will identify $\mathbb{R} \times \mathbb{R}^d$ with $\mathbb{R}^{d+1}$, and the $t$ coordinate with the $x_0$ coordinate (taking the coordinate system of $\mathbb{R}^{d+1}$ to run from $(x_0, x_1, \ldots, x_d)$). All displayed matrices will have indices running from $0, \ldots, d$ and the rows and

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column aligned accordingly. We also use the short hand \( \partial_i = \partial_{x_i} \). We will let \( m \) denote the diagonal matrix corresponding to the Minkowski metric:

\[
(1.1)\quad m := \begin{pmatrix}
-1 & & & \\
& 1 & & \\
& & 1 & \\
& & & \ddots
\end{pmatrix}
\]

Note that with matrix multiplication, \( m \cdot m = \delta \) (we use the Kronecker \( \delta \) symbol for the matrix with diagonal entries “1”, i.e. the identity matrix).

Notice that with the matrix \( m \) we can write the \( \Box \) operator as

\[
(1.2)\quad \Box \phi = \sum_{i,j=0}^{d} \partial_i (m_{ij} \partial_j \phi).
\]

Of principle importance in this discussion is the following object:

**1.3 Definition**

Let \( \phi \in C^2(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}) \) be arbitrary; its energy momentum tensor \( Q \) is the matrix-valued function given by

\[
Q_{ij} = \partial_i \phi \partial_j \phi - \frac{1}{2} m_{ij} \left( \sum_{k,\ell} m_{k\ell} \partial_k \phi \partial_\ell \phi \right), \quad i, j \in \{0,\ldots,d\}.
\]

As we saw on Homework 4 of this course, the energy momentum tensor enjoys the following divergence property, which relates it to the wave equation \([0.1]\).

**1.4 Lemma**

For any \( \phi \in C^2(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}) \):

\[
\sum_{i,j} \partial_i (m_{ij} Q_{jk}) = \Box \phi \partial_k \phi.
\]

Now, taking \( X = (X_0, X_1, \ldots, X_d) \) an arbitrary (smooth) vector field, we can define the vector field with components

\[
(1.5)\quad (X) J_i := \sum_{j,k} m_{ij} Q_{jk} X_k.
\]
The vector field $(X)\mathcal{J}$ is called the *energy-momentum current* associated to $X$. Its (space-time) divergence can be computed to be

\[
\sum_{i=0}^{d} \partial_{i}(X)\mathcal{J}_{i} = \sum_{i,j,k} \partial_{i}(m_{ij}Q_{jk}X_{k})
\]

\[
= \sum_{i,j,k} \partial_{i}(m_{ij}Q_{jk})X_{k} + \sum_{i,j,k} Q_{jk}m_{ji}\partial_{i}X_{k}
\]

The first term we can simplify using [Lemma 1.4](#) For the second term we observe that $Q$ is a symmetric matrix, so

\[
\sum_{i,j,k} Q_{jk}m_{ij}\partial_{i}X_{k} = \sum_{i,j,k} Q_{jk} \cdot \frac{1}{2} (m_{ji}\partial_{i}X_{k} + m_{ki}\partial_{i}X_{j}).
\]

### 1.6 Definition

Given the vector field $X$, its *deformation tensor* is the quantity

\[
^{(X,0)}\pi_{jk} := \sum_{i=0}^{d} \left( m_{ji}\partial_{i}X_{k} + m_{ki}\partial_{i}X_{j} \right).
\]

In this notation we conclude

\[
(1.7) \sum_{i=0}^{d} \partial_{i}(X)\mathcal{J}_{i} = \square \phi(X \cdot \partial \phi) + \frac{1}{2} \sum_{j,k} Q_{jk}^{(X,0)}\pi_{jk}.
\]

This immediately implies

### 1.8 Lemma

Let $\phi$ be a solution to the wave equation. Then for any vector field $X$ with vanishing deformation tensor, the current $(X)\mathcal{J}$ is divergence free.

### 1.9 Corollary

If $\Omega$ is an open (space-time) region with piecewise $C^{1}$ boundaries, $\phi$ solves the wave equation, and $X$ has vanishing deformation tensor, then

\[
\int_{\partial \Omega}^{(X)\mathcal{J} \cdot \vec{n}} \, dS = 0.
\]

Here $\vec{n}$ refers to the unit outward normal vector field along $\partial \Omega \subset \mathbb{R} \times \mathbb{R}^{d}$. 

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Corollary 1.9 is in fact an instance of Noether’s theorem.

Noether’s theorem states, roughly, that associated to each symmetry of a system is a corresponding conservation law. In classical mechanics, time-translation symmetry corresponds to conservation of energy, spatial translation symmetry correspond to conservation of momentum, and rotational symmetry corresponds to conservation of angular momentum.

Here, the statement that \((X,0) \pi \equiv 0\) is the assertion that \(X\) is a symmetry of the wave equations. The conclusion that a hypersurface integral vanishes is the conservation law. We will see better how this translates later when we discuss applications.

For our immediate applications, it suffices to observe that any vector field with constant coefficients automatically has vanishing deformation tensor; one should keep in mind that in further developments of the theory other vector fields can also come into play.

2. Space-time geometry

We make a few basic definitions; these should be familiar to readers with some experience in special relativity.

2.1 Definition
A vector \(X\) is said to be

- \textit{time-like} if \(X^T m X < 0\).
- \textit{space-like} if \(X^T m X > 0\).
- \textit{light-like} or \textit{null} if \(X^T m X = 0\).

2.2 Example
The vector \((1,0,\ldots,0)\) is time-like, but the vectors \((0,0,\ldots,0,1,0,\ldots,0)\) is space-like. The vector \((1,1,0,0,\ldots,0)\) is null.

We can similarly characterize hypersurfaces by their unit normals.

2.3 Definition
A hypersurface \(\Sigma \subset \mathbb{R} \times \mathbb{R}^d\) is said to be

- \textit{space-like} (or sometimes, co-time-like) if \(\vec{n}\) is time-like everywhere along \(\Sigma\).
- \textit{time-like} (or sometimes, co-space-like) if \(\vec{n}\) is space-like everywhere along \(\Sigma\).
- \textit{light-like} or \textit{null} if \(\vec{n}\) is null.
Here $\vec{n}$ is a unit normal vector field along $\Sigma$. □

2.4 Example
The hypersurface $\{x_0 = 0\}$ is space-like, the hypersurface $\{x_i = 0\}$ for any $i \in \{1, \ldots, d\}$ is time-like, and the hypersurface $\{x_0 + x_1 = 0\}$ is null. □

A key proposition that gives the usefulness of the energy method is the following algebraic property (which you are asked to prove on HW5):

2.5 Proposition
If $X, Y$ are two time-like vectors, such that $X \cdot Y > 0$. Then

$$(X)J \cdot Y \leq 0$$

with equality at a point $(t, x)$ if and only if $(t, x)$ is a critical point of $\phi$. □

We illustrate the proposition with a specific example:

2.6 Example
Let $X = Y = \partial_t$ be the vector corresponding to $(1, 0, 0, \ldots, 0)$. Clearly $X^T mX = m_{00} = -1 < 0$ so $X$ is time-like. $X \cdot Y = 1 > 0$. We can compute

$$(X)J \cdot Y = \sum_{i,j,k} m_{ij} Q_{jk} X_k Y_i$$

$$= \sum_j m_{0j} Q_{j0}$$

$$= m_{00} Q_{00}$$

$$= -\left[ \partial_0 \phi \partial_0 \phi - \frac{1}{2} m_{00} \left( \sum_{i=1}^d (\partial_i \phi)^2 - (\partial_0 \phi)^2 \right) \right]$$

$$= -\frac{1}{2} \sum_{i=0}^d (\partial_i \phi)^2.$$

The expression is the negative of a sum of squares, and hence is non-positive, and vanishes precisely when $\partial_i \phi = 0$ for every $i$. □

The following is the main technical result of these notes.

2.7 Theorem
Let $\Omega$ be an open space-time region with piecewise $C^1$ boundary. Assume that $\partial \Omega = \Sigma_+ \cup \Sigma_-$ where

- Both $\Sigma_+$ and $\Sigma_-$ are space-like.
- On $\Sigma_+$ we have $\vec{n} \cdot (\partial_t) > 0$;
• and on $\Sigma_-$ we have $\vec{n} \cdot (\partial_t) < 0$.

If $\phi \in C^2(\overline{\Omega}; \mathbb{R})$ solves $\Box \phi = 0$ and $\nabla(t, x) \phi \equiv 0$ along $\Sigma_+$ (resp. $\Sigma_-$), then $\nabla(t, x) \phi \equiv 0$ along $\Sigma_-$ (resp. $\Sigma_+$).

**Proof** From Corollary 1.9 we have that

$$\int_{\Sigma_+} (\partial_t) \cdot \vec{n} \, d\mathcal{S} + \int_{\Sigma_-} (\partial_t) \cdot \vec{n} \, ds = 0.$$  

By assumption the first term vanishes as a consequence of Proposition 2.5, the proposition also implying that the second term in the sum has non-negative integrand. That the total integral vanishes implies that the integrand must also vanish, which using the proposition again implies the desired result.

3. **Local uniqueness for wave equation**

We now return to the initial value problem for the wave equation (0.1). One implication of Theorem 2.7 is the following.

3.1 **Theorem**

Let $\phi$ be a solution of (0.1). Suppose the initial data are such that $g \equiv 0 \equiv h$ on the set $B(0, r)$. Then $\phi \equiv 0$ on the set

$$\{(t, x) \mid |t| + |x| \leq r\}.$$  

**Proof** Define $\Sigma_0 = \{0\} \times B(0, r)$. For $\tau \in (0, r)$, we let

$$\Sigma_\tau = \left\{ \{\tau\} \times B(0, \frac{r - \tau}{2}) \right\} \cup \left\{ (t, x) \mid r - \frac{\tau + r}{2} t = |x|, \, t \in [0, \tau] \right\}.$$  

Let $\Omega$ be the frustum bounded below by $\Sigma_0$ and above by $\Sigma_\tau$. Quite clearly $\Sigma_0$ is a space-like hypersurface with $\vec{n} \cdot \partial_t < 0$. For $\Sigma_\tau$, we observe that the top of the frustum is a portion of the hypersurface $\{t = \tau\}$ and so is space-like with $\vec{n} \cdot \partial_t > 0$. For the sides of the frustum the unit outward normal can be found to be the unit vector in the direction of

$$\left( \frac{\tau + r}{2\tau}, \frac{x_1}{|x|}, \frac{x_2}{|x|}, \ldots, \frac{x_d}{|x|} \right).$$  

By virtue of the fact that $\tau < r$ we have $\tau + r > 2\tau$, this implies that $\vec{n}$ is time-like, and $\vec{n} \cdot \partial_t > 0$.

Applying Theorem 2.7 with $\Sigma_-$ being $\Sigma_0$ and $\Sigma_+$ being $\Sigma_\tau$ for any of the fixed $\tau \in (0, r)$, we conclude that along $\Sigma_\tau$, $\nabla(t, x) \phi \equiv 0$. 

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Finally, observe that
\[ \bigcup_{\tau \in (0,r)} \Sigma_\tau = \{(t,x) \mid t \in [0,r], t + |x| \leq r\} \]
we get that \( \nabla \phi \) vanishes identically on this set, and since \( \phi \) is therefore constant it must vanish by the initial conditions. We have the desired conclusion after treating the region with \( t < 0 \) with a time reflection. \hfill \square

We can extend the previous theorem to initial data that vanish on sets of general shapes. We introduce the following notations.

3.2 Definition
Given any open \( \Omega \in \mathbb{R}^d \), its \textit{domain of dependence} is the set
\[ \mathcal{D}(\Omega) = \bigcup_{(y,r)} \{(t,x) \in \mathbb{R} \times \mathbb{R}^d \mid |t| + |x-y| \leq r\} \]
where the union is taken over all \((y,r) \in \Omega \times \mathbb{R}_+\) such that \( B(y,r) \subset \Omega \).

Given any open \( \Omega \in \mathbb{R}^d \), its \textit{domain of influence} is the set
\[ \mathcal{I}(\Omega) = (\mathbb{R} \times \mathbb{R}^d) \setminus \mathcal{D}(\mathbb{R}^d \setminus \overline{\Omega}). \]

3.3 Proposition
A point \((t,x) \in \mathbb{R} \times \mathbb{R}^d\) is in \( \mathcal{I}(\Omega) \) if and only if there exists a point \( y \in \Omega \) such that the vector \((t,x-y)\) is time-like. \hfill \blacksquare

Proof Observe that the condition \((t,x-y)\) is time-like \( \iff y \in B(x,|t|) \). We work by proving the contrapositive: since \( B(x,|t|) \) is open, we have that \( B(x,|t|) \cap \Omega = \emptyset \) if and only if \( B(x,|t|) \subset \mathbb{R}^d \setminus \overline{\Omega} \), which is in turn equivalent to \((t,x) \in \mathcal{D}(\mathbb{R}^d \setminus \overline{\Omega})\) by definition. \hfill \square

3.4 Theorem
Let \( \phi \) be a solution of (0.1). Suppose the initial data are such that there exists some open \( \Omega \subset \mathbb{R}^d \) such that \( g = h = 0 \) on \( \Omega \). Then \( \phi \equiv 0 \) on \( \mathcal{D}(\Omega) \). \hfill \blacksquare

Proof Immediate consequence of \textbf{Theorem 3.1}. \hfill \square

This is precisely \textit{finite speed of propagation property}: indeed if \( \text{supp} \, g \cup \text{supp} \, h \subset \Omega \) for some \( \Omega \) open, then the above theorem implies \( \text{supp} \, \phi \subset \mathcal{I}(\Omega) \), which only contains points \((t,x)\) such that \( \text{dist}(x,\Omega) < |t| \).

A uniqueness statement is also available for solutions on bounded domains.
3.5 Theorem
Let $\Omega \subset \mathbb{R}^d$ be open, bounded, with $C^1$ boundary. If $\phi \in C^2([0, T] \times \overline{\Omega})$ solves the initial-boundary-value problem

\begin{align*}
\Box \phi &= 0 \\
\phi(0, x) &= \partial_t \phi(0, x) = 0, \quad x \in \Omega \\
\phi(t, y) &= 0, \quad t \in [0, T], y \in \partial\Omega
\end{align*}

Then $\phi \equiv 0$.

Proof Let $\Omega_\tau = (0, \tau) \times \Omega$. Apply Corollary 1.9 to $\Omega_\tau$ with $X = \partial_t$. The boundary $\partial\Omega_\tau$ has three pieces: $\{\tau\} \times \Omega$, $\{0\} \times \Omega$, and $[0, \tau] \times \partial\Omega$. On the very last portion, note that $\vec{n}$ is orthogonal to $\partial_t$, and so

\[
(\partial_t) \mathcal{J} \cdot \vec{n} = \sum_i Q_{10} n_i = \partial_t \phi \partial_n \phi.
\]

This quantity vanishes as along $[0, \tau] \times \partial\Omega$, by assumption $\phi \equiv 0$ and hence $\partial_t \phi \equiv 0$. Therefore we are left with

\[
\int_{[0, \tau] \times \Omega} (\partial_t) \mathcal{J} \cdot \vec{n} \, dS + \int_{\{\tau\} \times \Omega} (\partial_t) \mathcal{J} \cdot \vec{n} \, dS = 0.
\]

The first term vanishes by the initial condition. The integrand in the second term is signed by Proposition 2.5 and hence must identically vanish as the integral is zero. By the proposition again we have that $\nabla(t, x) \phi \equiv 0$ on $[\tau] \times \Omega$. As $\tau$ is arbitrary, this means that $\nabla(t, x) \phi \equiv 0$ on $(0, T) \times \Omega$, and hence is constant, and hence vanishes.

4. Propagation of regularity

This final section addresses something mentioned when we discussed the fundamental solution of the wave equation, that the solution seems to be less differentiable than the given initial data, with the situation getting worse and worse in higher and higher dimensions. It turns out that this can be ameliorated if one thinks of regularity not in the pointwise differentiability sense, but in the sense of energy integrals.

We illustrate this point by first writing down two “energy estimates”.

4.1 Proposition

Suppose $\phi$ solves (0.1) with $g, h \in C^\infty_c(\mathbb{R}^d)$. Then for every $t \in \mathbb{R}$ we have that

\[
\int_{\mathbb{R}^d} \left| \partial_t \phi(t, x) \right|^2 + \left| \nabla(x) \phi(t, x) \right|^2 \, dx = \int_{\mathbb{R}^d} |h|^2 + |\nabla g|^2 \, dx.
\]
This follows from applying Corollary 1.9 to the slab $[0, t] \times \mathbb{R}^d$ and using that by finite speed of propagation $x \mapsto \phi(t, x)$ has compact support for every $t$. □

4.2 Proposition
Suppose $\phi$ solves (0.1). Then for any $(t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $r \in \mathbb{R}_+$, we have that

$$\int_{B(x, r)} \left| \partial_t \phi(t, y) \right|^2 + \left| \nabla \phi(t, y) \right|^2 \, dy \leq \int_{B(x, r + |t|)} \left| h \right|^2 + \left| \nabla g \right|^2 \, dy.$$ ■

Proof The claim follows from applying Corollary 1.9 to the frustum given by $\Omega = \mathcal{D}(B(x, r + |t|)) \cap [0, t] \times \mathbb{R}^d$. Notice that by a direct computation, for the portion of $\partial \Omega$ that is the sloped sides of the frustum, we can check $(\partial_t) J \cdot \vec{n} \leq 0$ (similar to the proof of Theorem 3.1). □

Now, on homework 5 you are asked to show also the following result.

4.3 Lemma
If $\phi$ solves the wave equation, and if $X$ is any vector field with $(X, 0) \pi \equiv 0$, then the function $\psi = \sum_{i=0}^d X_i \partial_i \phi$ also solves the wave equation. ■

So the energy estimates in Proposition 4.1 and 4.2 would also apply, provided that $g, h$ are replaced by their correct counterparts. In particular, we have that for $\alpha$ any multi-index for $\mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \left| \partial_t \partial^\alpha \phi(t, x) \right|^2 + \left| \nabla \partial^\alpha \phi(t, x) \right|^2 \, dx = \int_{\mathbb{R}^d} \left| \partial^\alpha h \right|^2 + \left| \nabla \partial^\alpha g \right|^2 \, dx \quad (4.4)$$

in the case $g, h$ have compact support, or in the general case

$$\int_{B(x, r)} \left| \partial_t \partial^\alpha \phi(t, y) \right|^2 + \left| \nabla \partial^\alpha \phi(t, y) \right|^2 \, dy \leq \int_{B(x, r + |t|)} \left| \partial^\alpha h \right|^2 + \left| \nabla \partial^\alpha g \right|^2 \, dy. \quad (4.5)$$

In this sense, we have the informal statement that

A solution of the wave equation on $\mathcal{D}(\Omega)$ is as regular as its initial data in $\Omega$, provided we measure regularity in the energy sense.

This is one of the reasons why in the second course in this qualifying exam sequence you will be learning about Sobolev spaces, which will allow you to make the above informal statement more precise.
References

