Plane waves and radiation field

Willie Wong[*]

Based on commit 5dfb7e5 of 2017-11-06 13:29

Abstract A quick introduction to the notion of Friedlander's radiation field, and its connections to plane-wave solutions of the wave equations, using the explicit formulae available in three spatial dimensions.

We start by recalling the solution to the one dimensional wave equation: by the method of characteristics we have that every solution $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ of

$$-\partial^2_{tt}\phi + \partial^2_{xx}\phi = 0$$

can be decomposed as the sum of traveling waves

$$\phi(t, x) = \psi_+(x + t) + \psi_-(x - t) \tag{0.1}$$

where the functions $\psi_+, \psi_- : \mathbb{R} \to \mathbb{R}$ are determined (up to an additive constant) by the initial data. Physically, we may expect something similar for higher dimensional waves: that a solution to the linear wave equation

$$\Box\phi = 0 \tag{0.2}$$

on $\mathbb{R} \times \mathbb{R}^d$ can be represented as a superposition of "plane waves". (In this context, the fundamental solution, such as those given by the Kirchhoff or Poisson parametrices, can be thought of as being related to the description of the solution as a superposition by "spherical waves".)

1. Plane waves

By a plane wave solution to (0.2) we mean that a solution that can be factored (in the sense of function composition) through an affine map.

1.1 Definition
A solution $\phi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ to (0.2) is said to be a *plane wave solution* if there exists a function $\psi : \mathbb{R} \to \mathbb{R}$ and a linear function $A : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ such that $\phi = \psi \circ A$. ■

1.2 Proposition
If ϕ is a plane wave solution to (0.2) then either:

[*]Michigan State University, East Lansing, USA; wongwwy@member.ams.org

- there exists an unit vector $\omega \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$ and a function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t,x) = \psi(t + \omega \cdot x)$, or
- ϕ is an affine function on $\mathbb{R} \times \mathbb{R}^d$. ■

Proof By definition we can find some $\tilde{\psi}$ and A such that $\phi = \tilde{\psi} \circ A$. Since A is a linear function, we have that ∇A is constant. So we have $\partial^2_{tt}\phi = (\partial_t A)^2(\tilde{\psi}'') \circ A$ and $\triangle\phi = |\nabla A|^2(\tilde{\psi}'') \circ A$. Therefore (0.2) implies either $(\partial_t A)^2 = |\nabla A|^2$ or $\tilde{\psi}'' \equiv 0$. The latter case implies ϕ is an affine function, since it is a composition of an affine function $\tilde{\psi}$ with a linear function A. In the former case, observe that any linear function A can be written in the form

$$A(t,x) = \tau t + \omega \cdot x,$$

for some $\tau \in \mathbb{R}$ and $\omega \in \mathbb{R}^d$, and the derivative condition on A requires

$$\tau^2 = |\omega|^2.$$

If $\tau = 0$, then necessarily $\omega = 0$ and A is the constant map, in which case ϕ is the constant function and is also affine. If, on the other hand, $\tau \neq 0$, then after redefining $\tilde{\psi}$ by rescaling the function's inputs, we can assume, without loss of generality, that $\tau = 1$, and $|\omega| = 1$ as claimed in the first alternative. □

1.3 Lemma

Any affine function on $\mathbb{R} \times \mathbb{R}^d$ can be expressed as a superposition of functions of the form $\psi(t + \omega \cdot x)$. ■

Proof Let $e_1, \ldots, e_d$ be the standard unit vectors in $\mathbb{R}^d$ viewed as a subspace of $\mathbb{R} \times \mathbb{R}^d$, and let e_0 be the unit vector in the time direction. Observe that the set

$$\{e_0 - e_1, e_0 + e_1, e_0 + e_2, e_0 + e_3, \ldots, e_0 + e_d\}$$

forms a basis of $\mathbb{R} \times \mathbb{R}^d$. The claim follows. □

Because of Lemma 1.3, we can focus on only the decomposition of arbitrary solutions as superpositions of functions of the form $\psi(t + \omega \cdot x)$. We see that (0.1) is such a decomposition.

1.4 Definition

We say that a solution ϕ to (0.2) *admits a plane wave decomposition* if there exists a function $\psi : \mathbb{R} \times \mathbb{S}^{d-1} \to \mathbb{R}$ such that

$$\phi(t,x) = \int_{\mathbb{S}^{d-1}} \psi(t + \omega \cdot x; \omega)\, \mathrm{d}\omega. \tag{1.5}$$ ■

Note that assuming ψ is sufficiently regular, by differentiating under the integral sign any function of the form (1.5) is a solution to the wave equation.

For convenience, we will refer to the derivative with respect to the $\mathbb{R}$ factor of ψ as ψ', and the derivative with respect to the $\mathbb{S}^{d-1}$ factor as $\nabla\!\!\!/\psi$.

In view of this, the questions becomes (similar to the one dimensional case):

> Can we solve for ψ in terms of the initial data g, h of the wave equations? More precisely, given $g, h : \mathbb{R}^d \to \mathbb{R}$, can we find $\psi : \mathbb{R}\times\mathbb{S}^{d-1} \to \mathbb{R}$ such that
>
> $$g(x) = \int_{\mathbb{S}^{d-1}} \psi(\omega \cdot x; \omega)\, \mathrm{d}\omega, \text{ and} \tag{1.6a}$$
>
> $$h(x) = \int_{\mathbb{S}^{d-1}} \psi'(\omega \cdot x; \omega)\, \mathrm{d}\omega. \tag{1.6b}$$

This question is then that of the invertibility of certain “integral transformation” and its further development is the field of “integral geometry”. In physics, this is also related to the development of “twister theory”.

2. The radiation field

For convenience, let us restrict the discussion to three dimensions.

By the Strong Huygens Principle, we know that if the initial data $g(x) = \phi(0,x)$ and $h(x) = \partial_t\phi(0,x)$ have compact support, then the only points (t,x) at which $\phi(t,x) \neq 0$ are those such that can be joined to the support of g and h with a light-like vector. In other words, denoting by $K = \operatorname{supp} g \cup \operatorname{supp} h$, the “action” only takes place on the set

$$K^* := \left\{(\lambda, y - \lambda\omega) \in \mathbb{R}\times\mathbb{R}^3 \;\middle|\; y \in K, \lambda \in \mathbb{R}, \omega \in \mathbb{S}^2\right\}. \tag{2.1}$$

It is then natural to ask about the *asymptotic properties* of the solution on K^*, in particular, how does ϕ behave as $\lambda \nearrow +\infty$ (or $\searrow -\infty$) when y and λ are held fixed?

This can be answered in one way using the fundamental solution (Kirchhoff parametrix). Previously we have already shown that the solution decays *uniformly* like $|t|^{-1}$ for $|t| \geq 1$. Hence we know that

$$\lim_{\lambda\to\pm\infty} \phi(\lambda, y - \lambda\omega) = 0.$$

The question then is, what is

$$\lim_{\lambda\to\pm\infty} \lambda\phi(\lambda, y - \lambda\omega)?$$

For $\lambda > 0$, we can use the fundamental solution to write

$$\lambda\phi(\lambda, y-\lambda\omega) = \frac{1}{4\pi\lambda} \int\limits_{K\cap\partial B(y-\lambda\omega,\lambda)} g(z) + \lambda\partial_n g(z) + \lambda h(z)\, \mathrm{d}S(z). \tag{2.2}$$

As previously discussed, $K \cap \partial B(y-\lambda\omega, \lambda)$ admits a uniform area bound. Furthermore, we can define the set

$$\Pi_{y,\omega} := \{\text{the hyperplane through the point } y \text{ orthogonal to } \omega\}. \tag{2.3}$$

Since K is compact we have that the set

$$K \cap \partial B(y-\lambda\omega, \lambda) \to K \cap \Pi_{y,\omega}$$

uniformly as $\lambda \to \infty$. So this means

$$\lim_{\lambda\to\infty} \lambda\phi(\lambda, y-\lambda\omega) = \frac{1}{4\pi} \int\limits_{K\cap\Pi_{y,\omega}} \omega\cdot\nabla g(z) + h(z)\, \mathrm{d}S(z). \tag{2.4}$$

The right hand side of this expression is essentially the *Radon transform*[1] of the initial data.

Notice that on the right side of the equation (2.4), the integral is over the plane $\Pi_{y,\omega}$; for any $y' \in \Pi_{y,\omega}$, the planes $\Pi_{y,\omega}$ and $\Pi_{y',\omega}$ coincide. This implies that the dependence of the limit on the variable y is *only through the component* $y\cdot\omega$ which is constant (by definition) along $\Pi_{y,\omega}$.

2.5 Definition

Given a solution to the wave equation $\phi : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$, its *radiation field* is the function $\Phi : \mathbb{R}\times\mathbb{S}^2$ given by

$$\Phi(y\cdot\omega, \omega) = \lim_{\lambda\to\infty} \lambda\phi(\lambda, y-\lambda\omega) = \frac{1}{4\pi} \int\limits_{\Pi_{y,\omega}} \omega\cdot\nabla g(z) + h(z)\, \mathrm{d}S(z).$$

■

Interestingly, we now have *two* functions defined on $\mathbb{R}\times\mathbb{S}^2$ associated to a solution ϕ to the wave equation on $\mathbb{R}\times\mathbb{R}^3$: the first is its plane wave decomposition ψ, and the second is its radiation field Φ. Are the two functions perhaps related?

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[1] A very brief description of the Radon transform: let $\mathcal{P}$ denote the set of all hyperplanes of $\mathbb{R}^d$. Given a function $f : \mathbb{R}^d \to \mathbb{R}$, its Radon transform is the function $\mathfrak{r}f : \mathcal{P} \to \mathbb{R}$ given by $\mathfrak{r}f(\Pi) = \int_\Pi f(z)\, \mathrm{d}S(z)$. The Radon transform is related to *tomography* and its inversion is related to reconstruction of data from its cross-sectional tomography scans.

3. The punch line

Let us compute the radiation field in terms of the plane-wave decomposition. For simplicity of computation assume that $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{S}^2)$. We then want to compute

$$\lim_{\lambda\to\infty} \lambda\phi(\lambda, y - \lambda\omega) = \lim_{\lambda\to\infty} \int_{\mathbb{S}^2} \lambda\psi(\lambda + y\cdot\omega' - \lambda\omega\cdot\omega'; \omega')\,\mathrm{d}\omega'. \tag{3.1}$$

First, let's consider the convergence property of this integral. The integrand is

$$\lambda\psi(\lambda(1-\omega\cdot\omega') + y\cdot\omega'; \omega').$$

For large λ, the compact support of ψ implies that only when ω is very close to ω' (so that $\omega\cdot\omega' \approx 1$) would we be on the support of the integrand. More precisely, using that for $\omega' \in \mathbb{S}^2$ near ω we can approximate

$$1 - \omega\cdot\omega' \approx \frac{1}{2}\left|\omega - \omega'\right|^2$$

we see that outside the region where

$$\left|\omega - \omega'\right| \lesssim \frac{1}{\sqrt{\lambda}}$$

the integrand vanishes. This justifies the boundedness of the integral in (3.1). This also indicates how we can compute the limit.

Near the point ω we can view $\mathbb{S}^2$ as a graph over the tangent plane at ω, that is to say, we can look at the hemisphere centered at $\mathbb{S}^2$ as described by

$$\{(x, 1 - \sqrt{1-|x|^2}) \in \mathbb{R}^3 \mid x \in B(0,1)\}.$$

Based on what we computed in the previous section, we expect that the limit to compute is independent of the choice of y as long as $y\cdot\omega$ is the same; so we can assume that $y \parallel \omega$. This means that the integral can be rewritten as

$$\int_{B(0,1)} \lambda f(\lambda(1-\sqrt{1-|x|^2}) + \tilde{y}\sqrt{1-|x|^2}; x)\,\frac{1}{\sqrt{1-|x|^2}}\,\mathrm{d}x \tag{3.2}$$

where $\tilde{y} = y\cdot\omega$. Our analysis above indicates that the integrand is supported only when $|x| \le \frac{c}{\sqrt{\lambda}}$ for some constant c for all λ sufficiently large. This suggests performing the change of variables

$$\tilde{x} = \sqrt{\lambda}x \tag{3.3}$$

and rewrite the integral as

$$\int\limits_{B(0,c)} f(\lambda(1-\sqrt{1-\lambda^{-1}|\tilde{x}|^2})+\tilde{y}\sqrt{1-\lambda^{-1}|\tilde{x}|^2};\lambda^{-1/2}\tilde{x})\frac{1}{\sqrt{1-\lambda^{-1}|\tilde{x}|^2}}\,\mathrm{d}\tilde{x}. \tag{3.4}$$

Using the Taylor expansion

$$1-\sqrt{1-\lambda^{-1}|\tilde{x}|^2}=\frac{1}{2}\lambda^{-1}|\tilde{x}|^2+o(\lambda^{-1}).$$

We see that the function

$$\tilde{x}\mapsto f(\lambda(1-\sqrt{1-\lambda^{-1}|\tilde{x}|^2})+\tilde{y}\sqrt{1-\lambda^{-1}|\tilde{x}|^2};\lambda^{-1/2}\tilde{x})\frac{1}{\sqrt{1-\lambda^{-1}|\tilde{x}|^2}}$$

converges, as $\lambda\to\infty$, uniformly on on $B(0,c)$ to the function

$$\tilde{x}\mapsto f(\tilde{y}+\frac{1}{2}|\tilde{x}|^2;0).$$

Using that f has compact support again, we can write the integral as

$$\int\limits_{\mathbb{R}^2} f(\tilde{y}+\frac{1}{2}|x|^2;0)\,\mathrm{d}x=2\pi\int\limits_0^\infty f(\tilde{y}+s;0)\,\mathrm{d}s.$$

We conclude finally that

$$\Phi(y\cdot\omega,\omega)=\lim_{y\to\infty}\lambda\phi(\lambda,y-\lambda\omega)=2\pi\int\limits_{y\cdot\omega}^\infty\psi(s;\omega)\,\mathrm{d}s \tag{3.5}$$

or, alternatively

$$\psi(s;\omega)=-\frac{1}{2\pi}\Phi'(s,\omega). \tag{3.6}$$

With this final formula, we are able to compute the coefficients ψ in the plane wave decomposition in terms of the initial data g and h, through the formula (2.4).

3.7 Remark

A short note about the compact support assumptions: our computation of the radiation field leading to (2.4) assumes that g,h both have compact support. Note that this implies that tha radiation field Φ has compact support: for every fixed ω

and every λ sufficiently large, if $y \cdot \omega = \lambda$ than the plane $\Pi_{y,\omega}$ does not intersect the support of g and h, and hence the integral (2.4) yields 0. A posteriori this means that if we define $\psi(s;\omega)$ by the formula (3.6) then the function ψ will also have compact support, which justifies the computations leading up to the formula.

So what we have shown is that *given every* $g, h \in C_c^\infty(\mathbb{R}^3)$, *there exists exactly one* $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{S}^2)$ *such that the solution given by Kirchhoff's formula applied to* g, h *and the solution corresponding to* (1.5) *applied to* ψ *have the same radiation field*. While this gives a very appealing formula, the complete proof of its correctness is beyond the scope of this course; we refer interested students to [1]. ∎

References

[1] Sigurdur Helgason. *The Radon transform*. Second. Vol. 5. Progress in Mathematics. Boston, MA: Birkhäuser Boston Inc., 1999, pp. xiv+188. ISBN: 0-8176-4109-2.