Small Data GWP

1. Known Facts

- Form of nonlinearity: wave $F(\partial\phi)$
  KG $F(\phi, \partial\phi)$

  vanish to order 2 @ origin $|F| = O(|\phi|^2 + |\partial\phi|^2)$

- Dispersive Estimate:

$$\|f\|_{\mathcal{L}^\infty_{\frac{d}{2}+s}} \lesssim \|f\|_{W^{\lfloor \frac{d}{2}\rfloor+1,2}_s}$$

2. Strategy

Given ϕ_0, ϕ_1 with cpct support on $B_1(0)$

try to solve $\Box_m \phi = F$ on $\{t \geq 2\}$

by finding a fixed point of the mapping

$$\rho(\Phi) = U(t-2)\Phi_0 - \int_2^t U(t-s)\begin{pmatrix} 0 \\ F(s) \end{pmatrix} ds$$

Here $\Phi = \begin{pmatrix} \phi \\ \partial_t\phi \end{pmatrix}$, $\Phi_0 = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix}$, U the

sol'n op for linear wave eq. $F(s) = F \circ \Phi(s)$ appropriately.

Key Q: What fn space?

3. Idea

For formula to make sense, need $F(s) \in L^2$ at least.

Since F is quadratic at origin:

$$F(\phi, \partial\phi) = G(\phi, \partial\phi) \cdot \begin{pmatrix} \phi \\ \partial\phi \end{pmatrix}$$

with $G(\phi, \partial\phi) = O(|\phi| + |\partial\phi|)$

smooth. (Malgrange Preparation Theorem)

So: if $\phi, \partial\phi \in L^\infty \cap L^2 \Rightarrow F \in L^2$

LWP Case

Use Sobolev embedding $H^{\lfloor d/2 \rfloor + 1} \hookrightarrow L^\infty$

$\Rightarrow$ " $\phi, \partial\phi \in H^{\lfloor d/2 \rfloor + 1} \Rightarrow F \in H^{\lfloor \frac{d}{2} \rfloor + 1}$ "

GWP Case

Use (tty) dispersive estimate !!

Example: (Wave)

Suppose Energy Flux on Σ_τ are uniformly bounded $\Rightarrow$ $\forall\, \Sigma_\tau$

$$\tau^d \|T\phi\|^2_{W^{k,2}_{-1}} + \tau^{d-2} \|L\phi\|^2_{W^{k,2}_{-1}} < M.$$

Assume Also $\operatorname{supp}(\phi) \subseteq \{ t \geq |x|+1 \}$

Then by dispersive estimate

$$|\partial\phi| \lesssim \frac{M}{t^{\frac{d-1}{2}}}$$

$$\Rightarrow \quad F(\partial\phi) = G(\partial\phi)\cdot\partial\phi \quad \in L^2$$

w/

$$\|F(\partial\phi)\|_{L^2} \lesssim \frac{M}{t^{\frac{d-1}{2}}} \|\partial\phi\|_{L^2}$$

(Use that F vanish Quad. $\Rightarrow G(\partial\phi) = O(|\partial\phi|)$

$\Rightarrow \exists\, C$ s.t. $\forall\, |\partial\phi| \leq M$

$$|G(\partial\phi)| \leq C|\partial\phi| \;)$$

4. Return to Strategy

① to make sense of $\rho(\Phi)$, need the energy flux for $\underbrace{L \cdots L}_{k\text{-fold}}\phi$ to be bounded

② To use contraction mapping, need

a. "boundedness of energy flux for Φ"

$\Rightarrow$ "boundedness of energy flux for $\rho(\Phi)$"

b. "ρ is a contraction"

5. Function Spaces

Short Hand Notation $\quad L^{[k]}$ = string of k L operators

$$= L^{i_1} L^{i_2} \cdots L^{i_k}$$

$$\|\phi\|^2_{E^k} = \sup_{t\geq 2} \left(\sum_{j=0}^{k} \| \partial L^{[j]}\phi \|^2_{L^2(\mathbb{R}^d)} + m \| L^{[j]}\phi \|^2_{L^2} \right)$$

$$\|\phi\|^2_{\mathcal{E}^k} = \sup_{\tau\geq 2} \left(\sum_{j=0}^{k} \tau^d \| \underset{\partial_t}{\overset{\shortparallel}{T}} L^{[j]}\phi \|^2_{\mathcal{L}^2_{-1}(\Sigma_\tau)} + \tau^{d-2} \| L^{[j+1]}\phi \|^2_{\mathcal{L}^2_{-1}} + m\, \tau^d \| L^{[j]}\phi \|^2_{\mathcal{L}^2_0} \right)$$

Commutator Estimate

$$[\partial_t, L^i] = \partial_{x^i}$$

$$[\partial_{x^i}, L^j] = \delta_{ij}\, \partial_t$$

$$\Rightarrow \quad \underbrace{L \cdots L}_{k} \partial = \sum_{j=0}^{k} \partial \underbrace{L \cdots L}_{j}$$

Ex

$$L^1 L^2 \partial_t = L^1 \partial_t L^2 - L^1 [\partial_t, L^2]$$

$$= \partial_t L^1 L^2 - [\partial_t, L^1] L^2 - L^1 [\partial_t, L^2]$$

$$= \partial_t L^1 L^2 - \partial_{x^1} L^2 - L^1 \partial_{x^2}$$

$$= \partial_t L^1 L^2 - \partial_{x^1} L^2 - \partial_{x^2} L^1 \qquad ([\partial_{x^2}, L^1] = 0)$$

Energy Bounds

$t \geq 2$

$$m \| L^{[k]} \phi \|^2_{L^2(\mathbb{R}^d)} \leq \|\phi\|^2_{E^k}$$

$$\| L^{[k]} \partial\phi \|^2_{L^2(\mathbb{R}^d)} \lesssim \sum_{j=0}^{k} \| \partial L^{[j]} \phi \|^2_{L^2(\mathbb{R}^d)}$$

$$\lesssim \|\phi\|_{E^k}$$

Pointwise Bounds

(with a light cone, $\operatorname{supp}(\phi) \subseteq \{ t \geq |x| + 1 \}$)

$m > 0$: $$\| L^{[k]} \phi \|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{t^{d/2}} \|\phi\|_{\mathcal{E}^{k + \lfloor \frac{d}{2} \rfloor + 1}}$$

$m=0$: $\| \partial L^{[K]} \phi \|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{t^{\frac{d-1}{2}}} \| \phi \|_{\mathcal{E}^{K+\lfloor \frac{d}{2} \rfloor + 1}}$

6. Estimates of nonlinearity

$$L^{[K]} F(\phi, \partial\phi) = \sum F^{(p)}(\phi, \partial\phi) \cdot L^{[K_1]}\phi \cdots L^{[K_j]}\phi \cdot L^{[K_{j+1}]}\partial\phi \cdots L^{[K_p]}\partial\phi$$

Pigeonhole: at most one of $K_1 \cdots K_p$ is $\geq \lceil \frac{K}{2} \rceil$

Schematically:

$$L^{[K]} F(\phi, \partial\phi) = G \cdot L^{[\leq K]} \begin{pmatrix} \phi \\ \partial\phi \end{pmatrix}$$

G: smooth fn of up to $\lfloor \frac{K}{2} \rfloor$ derivatives of $\phi, \partial\phi$, by L operators

Since F vanishes quadratically,

$G(0) = 0$

Prop for any fixed K, $\exists$ a increasing fn H with $H(s) = O(s)$ as $s \to 0$ s.t. $\forall\ 0 \leq j \leq K$, $\forall\ t \geq 2$

$$\| L^{[j]} F \|_{L^2(\mathbb{R}^d)} \leq H\left(\| \phi \|_{\mathcal{E}^{\lfloor \frac{K}{2} \rfloor + \lfloor \frac{d}{2} \rfloor + 1}} \cdot t^{\frac{1-d}{2}} \right) \| \phi \|_{E^K}.$$

Provided $\operatorname{supp}(\phi) \subseteq \{ t \geq |x| + 1 \}$

7. $d \geq 4$ GWP

THEOREM Let $d \geq 4$. $\Box_m \phi = F$ has unique, global solutions if initial data is supported on $B_1(0)$,

and $\|\phi_0\|_{H^{d+2}} + \|\phi_1\|_{H^{d+1}}$ is sufficiently small.

Pf

$\boxed{1}$ Homogeneous part: let $\mathring{\phi}$ solve $\Box_m \mathring{\phi} = 0$ with data (ϕ_0, ϕ_1) (so $\mathring{\Phi} = U(t-2)\Phi_0$)

$\Rightarrow \exists\ C > 0$ s.t.

$$\|\mathring{\phi}\|_{E^{d+1}} + \|\mathring{\phi}\|_{\mathcal{E}^{d+1}} \leq C\left(\|\phi_0\|_{H^{d+2}} + \|\phi_1\|_{H^{d+1}}\right)$$

This follows because by energy conservation for the homogeneous eq.

$$(LHS)^2 \lesssim \sum_{J=0}^{d} \int_{\mathbb{R}^d} |\partial_t L^J \phi|^2 + |\nabla L^J \phi|^2 + m|L^J\phi|^2 \, dx \Big|_{t=2}$$

using the eq to convert $\partial_t \partial_t \to \Delta$

and using that $t=2$, $|x|<1$ on $\operatorname{supp}(\phi_0, \phi_1)$

$$RHS \lesssim \|\phi_0\|^2_{H^{d+2}} + \|\phi_1\|^2_{H^{d+1}}$$

$\boxed{2}$ If $\phi \in \mathcal{E}^{d+1} \cap E^{d+1}$ & $\operatorname{supp}(\phi) \subseteq \{t \geq 2, t \geq |x|+1\}$

Then if $J \leq d+1$

$$\|L^{[J]} F\|_{L^2(\mathbb{R}^d)} \leq H\left(\|\phi\|_{\mathcal{E}^{\lfloor \frac{d+1}{2} \rfloor + \lfloor \frac{d}{2} \rfloor + 1}} \cdot t^{\frac{1-d}{2}}\right) \cdot \|\phi\|_{E^J}$$

Using that $\lfloor \frac{d+1}{2} \rfloor + \lfloor \frac{d}{2} \rfloor = d$, we see

$$\|L^{[J]} F\|_{L^2(\mathbb{R}^d)} \leq H\left(\|\phi\|_{\mathcal{E}^{d+1}} \cdot t^{\frac{1-d}{2}}\right) \cdot \|\phi\|_{E^d}$$

Using that $H(s) = O(s)$ as $s \to 0$

Conclude:

$\exists\, C > 0$ s.t. $\forall\, \phi$ with

$$\|\phi\|_{\mathcal{E}^{d+1}} < 1, \quad \forall\, J \le d+1$$

$$\|L^{[J]} F\|_{L^2} \le \frac{C \cdot \|\phi\|_{\mathcal{E}^{d+1}} \|\phi\|_{E^{d+1}}}{t^{\frac{d-1}{2}}}$$

$\boxed{3}$ Let

$$\psi = \text{first component of } \rho\begin{pmatrix}\phi \\ \partial_t \phi\end{pmatrix}$$

$$\begin{pmatrix}\psi \\ \partial_t \psi\end{pmatrix} = U(t-\tau)\begin{pmatrix}\phi_0 \\ \phi_1\end{pmatrix} - \int_\tau^t U(t-s)\begin{pmatrix}0 \\ F(s)\end{pmatrix} ds$$

Claim $\exists\, \varepsilon_0, \varepsilon > 0$ s.t., if $\operatorname{supp}(\phi) \subseteq \{t \ge |x|+1\}$

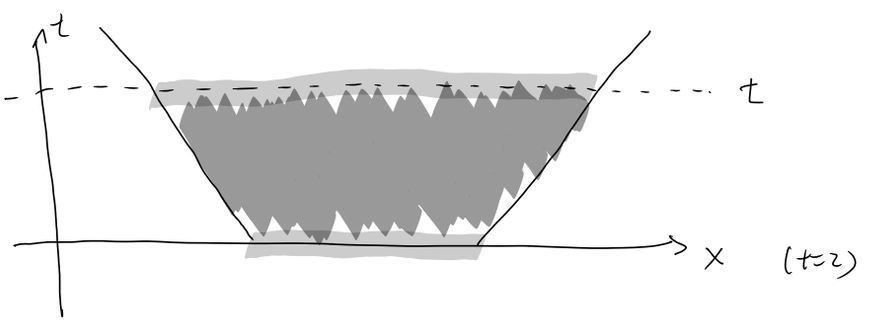
$$\left\{\begin{array}{l} \|\phi_0\|_{H^{d+2}} + \|\phi_1\|_{H^{d+1}} < \varepsilon_0 \\ \|\phi\|_{\mathcal{E}^{d+1}}, \|\phi\|_{E^{d+1}} < \varepsilon \end{array}\right. \Rightarrow \|\psi\|_{\mathcal{E}^{d+1}}, \|\psi\|_{E^{d+1}} < \varepsilon$$

& $\operatorname{supp}(\psi) \subseteq \{t \ge |x|+1\}$

Support condition: follows from finite speed of propagation.

E^{d+1}: Apply energy estimate to $L^{[J]}\psi$,

$J \le d+1$



ψ solves
$$\left\{\begin{array}{l} \Box_m \psi = F(\phi, \partial\phi) \\ \psi(\tau, x) = \phi_0(x) \\ \partial_t \psi(\tau, x) = \phi_1(x) \end{array}\right.$$

Energy estimate $\left(E^{d+1} = \sup_{t \geq 2} \sqrt{\text{flux through } t} \right)$

$$\Rightarrow \quad \|\psi\|_{E^{d+1}} \leq \|\mathring{\Phi}\|_{E^{d+1}} + \sum_{J=0}^{d+1} \int_2^\infty \|L^{[J]} F(\Phi)\|_{L^2} \, dt$$

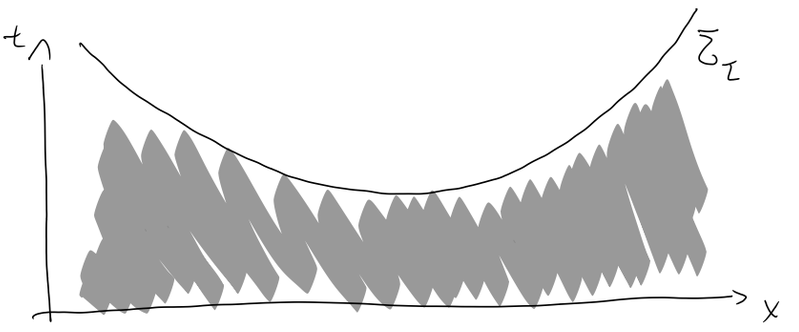
$$\leq C\varepsilon_0 + C' \int_2^\infty \frac{\varepsilon^2}{t^{\frac{d-1}{2}}} \, dt$$

$$\leq C\varepsilon_0 + C'' \varepsilon^2 \qquad \left(d \geq 4, \text{ this is integrable} \right)$$

$\Rightarrow$ provided $\varepsilon, \varepsilon_0$ chosen so that

$$C\varepsilon_0 + C''\varepsilon^2 < \varepsilon \quad \Rightarrow \text{ this energy controlled!!}$$

$\underline{\mathcal{E}^{d+1}}$



$$\left(\mathcal{E}^{d+1} \right)^2 = \sup_{\tau \geq 2} \text{ Flux through } \Sigma_\tau \text{ of } L^{[J]}\psi, \quad J \leq d+1$$

$$\text{Flux through } \Sigma_\tau \; L^{[J]}\psi \leq \text{initial data} + \int L^{[J]} F \cdot \partial_t L^{[J]} \psi$$

$$\int L^{[J]} F \cdot \partial_t L^{[J]} \psi \leq \int_{t \geq 2} |L^{[J]} F| \, |\partial_t L^{[J]} \psi|$$

$$\leq \int_2^\infty \|L^{[J]} F\|_{L^2} \underbrace{\|\partial_t L^{[J]} \psi\|_{L^2}}_{\leq \|\psi\|_{E^{d+1}}} \, dt$$

$$\leq \varepsilon \cdot \int_2^\infty \|L^{[J]} F\|_{L^2} \, dt$$

$$\leq C'' \varepsilon^3$$

data $\leq C^2 \varepsilon_0^2$

$\Rightarrow$ if $\varepsilon, \varepsilon_0$ chosen so that

$$C^2 \varepsilon_0^2 + C'' \varepsilon^3 \leq \varepsilon^2 \Rightarrow \text{can control this energy}$$

$\boxed{4}$ If $\varepsilon_0, \varepsilon$ sufficiently small $\Rightarrow$ this $\phi \mapsto \psi$ is contraction mapping.

Use that

$$F(\phi, \partial\phi) - F(\tilde{\phi}, \partial\tilde{\phi})$$

$$= \int_0^1 F'\left(s\phi + (1-s)\tilde{\phi},\ s\partial\phi + (1-s)\partial\tilde{\phi}\right) \cdot \begin{pmatrix} \phi - \tilde{\phi} \\ \partial\phi - \partial\tilde{\phi} \end{pmatrix} ds$$

$L^{[5]}\left(F(\phi, \partial\phi) - F(\tilde{\phi}, \partial\tilde{\phi})\right)$ has either

$$\begin{cases} \text{most } L \text{ derivatives on } \begin{pmatrix} \phi - \tilde{\phi} \\ \partial\phi - \partial\tilde{\phi} \end{pmatrix} \\ \text{or} \\ \text{most } L \text{ on } \left(s\phi + (1-s)\tilde{\phi},\ s\partial\phi + (1-s)\partial\tilde{\phi}\right) \end{cases}$$

<u>Case 1</u>

$$\| L^{[5]}(F - \tilde{F}) \|_{L^2} \leq H\left(\| s\phi + (1-s)\tilde{\phi} \|_{\mathcal{E}^{d+1}} \cdot \frac{1}{t^{\frac{d-1}{2}}} \right) \cdot \| \phi - \tilde{\phi} \|_{\mathcal{E}^{d+1}}$$

$[H(s) = O(s)]$

<u>Case 2</u>

$$\| L^{[5]}(F - \tilde{F}) \|_{L^2} \leq H'\left(\| s\phi + (1-s)\tilde{\phi} \|_{\mathcal{E}^{d+1}} \cdot \frac{1}{t^{\frac{d-1}{2}}} \right) \cdot \| \phi - \tilde{\phi} \|_{\mathcal{E}^{d+1}} \frac{1}{t^{\frac{d-1}{2}}} .$$

$[H'(s) = O(1)]$

$$\| s\phi + (1-s)\tilde{\phi} \|_{E^{d+1}}$$

$\diamond$

8. $d=3$ KG

<u>THM</u> also holds for $\Box_m \phi = F(\phi, \partial\phi)$ with $m>0$ when $d=3$, and use $d+3$ derivatives

<u>Key idea</u>: We can get KG decay like $t^{-\frac{d}{2}}$ if we use 1 more derivative.

<u>However</u>: to implement this we need to use, simultaneously, L derivatives <u>and</u> ∂ derivatives.

(For technical Reasons)

9. $d \leq 3$ Wave is False!!

Consider $\Box \phi = (\partial_t \phi)^2$ $\quad d=3$

Let $\eta = r\phi$, assume ϕ is rotationally symmetric

then
$$\Box \phi = \frac{1}{r}(\partial_t + \partial_r)(\partial_t - \partial_r)\eta$$

$\Rightarrow$ eq for η:

$$\frac{1}{r}(\partial_t + \partial_r)(\partial_t - \partial_r)\eta = \frac{1}{r^2}\left(\tfrac{1}{2}(\partial_t + \partial_r)\eta + \tfrac{1}{2}(\partial_t - \partial_r)\eta\right)^2$$

Choose ϕ_0 supported on $r \in [1,2]$

choose ϕ_1 s.t. $\phi_1 = -\frac{1}{r}\partial_r(r\phi_0)$ supported on $r \in [1,2]$

$$\Rightarrow (\partial_t + \partial_r)\eta \Big|_{t=0} = 0$$

$$\Rightarrow (\partial_t + \partial_r)\left[(\partial_t - \partial_r)\eta\right] = \frac{1}{4r}\left[(\partial_t - \partial_r)\eta\right]^2$$

$$\Rightarrow \frac{1}{(\partial_t - \partial_r)\eta(s, r+s)} - \frac{1}{(\partial_t - \partial_r)\eta(0,r)} = \frac{1}{4}\ln\left(\frac{r+s}{r}\right)$$

@ any point where
$$(\partial_t - \partial_r)\eta(0,r) < 0$$

$\exists$ some $s>0$ s.t.

$$(\partial_t - \partial_r)\eta(s, r+s) = \infty \quad !!$$